ON THE GENERALITY OF METHODS OF MATHEMATICAL PHYSICS AND NUMERICAL ANALYSIS FOR SOME BOUNDARY VALUE PROBLEMS-I

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Abstract

We consider the problems of creation the convergence difference schemes and numerical realization for the estimate of order of arithmetic operations needed for finding an approximate solution of Dirichlet problem for biharmonic equation in multidimensional cube. In this paper we consider and analyse one-dim case.

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1 Introduction

This study represents an ideological breakthrough that is essentially manifested when considering multidimensional boundary value problems (BVPs). In general, there are two directions: (1)The possibility to study BVPs by applying methods of mathematical physics together with methods of numerical analysis, which requires creation of new type functionals and determination of the properties of corresponding operators; (2) The possibility of effective numerical realization of generalized solutions or relatively less smooth boundary value problems.

As a first step, to see the effect of the method,we consider very simple 1D BVPs of second order operator. The results are compared with the results of an exact solution and direct application of classical finite difference method.

2 BVP-1: Second Order ODE

Consider the following BVP

$$\begin{aligned} &u'' = f, \\ &u|_{\partial\Omega} = 0, \\ \text{with } &u := u(x), \quad f := f(x) := 12x^2 - 12x + 2, \quad x \in \Omega := (0,1). \end{aligned} \tag{1}$$

3 Solution of the BVP-1

The exact solution of equation (1) is

$$u_e = x^2(1-x)^2 \tag{2}$$

The approximate solution is found by applying variational and numerical methods together. After some variational manupulations (see for example [1]) the approximate functional related to BVP-1 is derived for $i = \overline{1, n-1}$ as:

$$\int_\Omega [(u')^2 + 2fu]dx \cong \tilde{I}(\cdots, u_i, \cdots), \tag{3}$$

Let $\int_\Omega (u')^2 dx \cong \tilde{I}_L(\cdots, u_i, \cdots)$, $-\int_\Omega 2fudx \cong \tilde{I}_R(\cdots, u_i, \cdots)$. Then the corresponding Euler equation of BVP-1 requires that

$$\frac{\partial \tilde{I}_L}{\partial u_i} \cong \frac{\partial \tilde{I}_R}{\partial u_i}. \tag{4}$$

From now on, techniques of numerical analysis is to be applied to get algebraic system of equations from (4). Ω region can be discretised by dividing in n parts with a uniform increment of h, as $h = 1/n = x_{i+1} - x_i, (i = \overline{0, n-1})$. The left hand side integral given in (4) is derived by selecting 2-point central difference stencil for the first order derivative and covering the region Ω by passing through each inner point of the region $x_i (i = \overline{1, n-1})$. So

$$2h \sum_{i=1}^{n-1} \left(\frac{u_{i+1} - u_{i-1}}{2h} \right)^2 \cong 2 \int_0^1 (u')^2 dx - \int_0^h (u')^2 dx - \int_{(n-1)h}^{nh} (u')^2 dx. \tag{5}$$

From this relation the left hand side integral given in (4) can be derived as

$$\int_0^1 (u')^2 dx \cong \tilde{I}_L(\cdots, u_i, \cdots) := h\sum_{i=1}^{n-1}\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)^2 + \frac{1}{2}\int_0^h (u')^2 dx + \frac{1}{2}\int_{(n-1)h}^{nh} (u')^2 dx, \tag{6}$$

where the integrals near to the boundary points can be approximated by taking 3-point one sided derivative stencil for the end points and the integrals are approximated by applying closed trapezoidal rule as follows [2]

$$\begin{aligned}
&\int_0^h (u')^2 dx \cong h\cdot\frac{(u_0')^2+(u_1')^2}{2}\\
&u_0' = \frac{1}{2h}(-3u_0+4u_1-u_2),\\
&u_1' = \frac{u_2-u_0}{2h},\\
&\int_{(n-1)h}^{nh} (u')^2 dx \cong h\cdot\frac{(u_{n-1}')^2+(u_n')^2}{2},\\
&u_{n-1}' = \frac{u_n-u_{n-2}}{2h},\\
&u_n' = \frac{1}{2h}(3u_n-4u_{n-1}+u_{n-2}).
\end{aligned} \tag{7}$$

The right hand side integral given in (4) can be calculated by the open trapezoidal rule as

$$-2\int_0^1 fu dx \cong \tilde{I}_R(\cdots, u_i, \cdots) := -2h\sum_{i=1}^{n-1}(fu)_i \tag{8}$$

Now, as required by (4), the partial derivative of the left and right hand side approximate integrals given in (6) and (8) respectively can be taken, the results are given below

$$
\begin{aligned}
&\frac{\partial \tilde{I}_L}{\partial u_i} = \frac{1}{h}\left(u_i - \frac{1}{2}u_{i+2} - \frac{1}{2}u_{i-2}\right), \quad (i = \overline{3, n-3}),\\
&\frac{\partial \tilde{I}_L}{\partial u_1} = \frac{1}{h}\left(\frac{5}{2}u_1 - \frac{1}{2}u_2 - \frac{1}{2}u_3\right),\\
&\frac{\partial \tilde{I}_L}{\partial u_2} = \frac{1}{h}\left(-\frac{1}{2}u_1 + \frac{5}{4}u_2 - \frac{1}{2}u_4\right),\\
&\frac{\partial \tilde{I}_L}{\partial u_{n-2}} = \frac{1}{h}\left(-\frac{1}{2}u_{n-4} + \frac{5}{4}u_{n-2} - \frac{1}{2}u_{n-1}\right),\\
&\frac{\partial \tilde{I}_L}{\partial u_{n-1}} = \frac{1}{h}\left(-\frac{1}{2}u_{n-3} - \frac{1}{2}u_{n-2} + \frac{5}{4}u_{n-1}\right),\\
&\frac{\partial \tilde{I}_R}{\partial u_i} = -2hf_i.
\end{aligned}
\tag{9}
$$

Carefull consideration of the algebraic system given in (9) together with (4), reveals that the system can be divided into two subsytem for odd and even indices. Then the whole system $(n-1)\times(n-1)$ can be solved simply by reducing it to (4×4) system as follows.

Let us consider when $n = 2m$; let $u_\alpha = \nu_\alpha, u_{2m-\alpha} = w_\alpha (\alpha = 1, 2.)$ are parameters and $-2h^2 f(x_i) = \varphi_i (i = \overline{0, 2m})$. Thus we have the following steps:

1. Let in (9a) the indices are odd and by using (9f), we have:

$$
\begin{aligned}
&u_{2i+1} = \frac{1}{2}u_{2i-1} + \frac{1}{2}u_{2i+3} + \varphi_{2i+1} \quad (i = \overline{1, m-2}) \Rightarrow u_3 =\\
&\frac{1}{2}u_5 + \frac{1}{2}\nu_1 + \varphi_3, \quad \varphi_3 = F_3,\\
&u_5 = \frac{2}{3}u_7 + \frac{1}{3}\nu_1 + F_5, \quad F_5 = \frac{4}{3}\left(\frac{1}{2}F_3 + \varphi_5\right), \cdots,\\
&u_{2k+1} = \frac{k}{k+1}u_{2k+3} + \frac{1}{k+1}\nu_1 + F_{2k+1},\\
&F_{2k+1} = \frac{2k}{k+1}\left(\frac{1}{2}F_{2k-1} + \varphi_{2k+1}\right).
\end{aligned}
\tag{10}
$$

Taking $k = m-2$ in (10a) and using (9e) immediately follows:

$$
-\frac{1}{2(m-1)}\nu_1 + \frac{4m-3}{2(m-1)}w_1 - \frac{1}{2}w_2 = \phi_1, \quad \phi_1 = \frac{1}{2}F_{2m-3} + \varphi_{2m-1}. \tag{11}
$$

2. Let in (9a) the indices are even and by using (9f), we analogously have:

$$\begin{aligned}
&u_{2i} = \frac{1}{2}u_{2i+2} + \frac{1}{2}u_{2i-2} + \varphi_{2i} \ \ (i = \overline{2, m-2}) \Rightarrow u_4 = \\
&\frac{1}{2}u_6 + \frac{1}{2}\nu_2 + \varphi_4, \quad \varphi_4 = F_4, \\
&u_6 = \frac{2}{3}u_8 + \frac{1}{3}\nu_2 + F_6, \quad F_6 = \frac{4}{3}\left(\frac{1}{2}F_4 + \varphi_6\right), \cdots, \\
&u_{2k+2} = \frac{k}{k+1}u_{2k+4} + \frac{1}{k+1}\nu_2 + F_{2k+2}, \\
&F_{2k+2} = \frac{2k}{k+1}\left(\frac{1}{2}F_{2k} + \varphi_{2k+2}\right).
\end{aligned} \tag{12}$$

Taking $k = m - 3$ in (12a) and using (9d) we have:

$$-\frac{1}{2(m-2)}\nu_2 - \frac{1}{2}w_1 + \frac{3m-4}{4(m-2)}w_2 = \phi_2, \quad \phi_2 = \frac{1}{2}F_{2m-4} + \varphi_{2m-2}. \tag{13}$$

3. Let again in (9a) the indices are odd and by using (9f), we have:

$$\begin{aligned}
&u_{2i+1} = \frac{1}{2}u_{2i-1} + \frac{1}{2}u_{2i+3} + \varphi_{2i+1} \ \ (i = \overline{1, m-2}) \Rightarrow u_{2m-3} = \\
&\frac{1}{2}u_{2m-5} + \frac{1}{2}w_1 + \varphi_{2m-3}, \quad \varphi_{2m-3} = \bar{F}_{2m-3}, \\
&u_{2m-5} = \frac{2}{3}u_{2m-7} + \frac{1}{3}w_1 + \bar{F}_{2m-5}, \quad \bar{F}_{2m-5} = \frac{4}{3}\left(\frac{1}{2}\bar{F}_{2m-3} + \varphi_{2m-5}\right), \cdots, \\
&u_{2m-2k-1} = \frac{k}{k+1}u_{2m-2k-3} + \frac{1}{k+1}w_1 + \bar{F}_{2m-2k-1}, \\
&\bar{F}_{2m-2k-1} = \frac{2k}{k+1}\left(\frac{1}{2}\bar{F}_{2m-2k+1} + \varphi_{2m-2k-1}\right).
\end{aligned} \tag{14}$$

Taking $k = m-2$ in (14a) and using (9b), we have the following relation:

$$-\frac{1}{2(m-1)}w_1 + \frac{4m-3}{2(m-1)}\nu_1 - \frac{1}{2}\nu_2 = \bar{\phi}_1, \quad \bar{\phi}_1 = \frac{1}{2}\bar{F}_3 + \varphi_1. \tag{15}$$

4. Let again in (9a) the indices are even and by using (9f), we have:

$$
\begin{aligned}
&u_{2i} = \frac{1}{2}u_{2i+2} + \frac{1}{2}u_{2i-2} + \varphi_{2i} \ \ (i = \overline{2, m-2}) \Rightarrow u_{2m-4} = \\
&\frac{1}{2}u_{2m-6} + \frac{1}{2}w_2 + \varphi_{2m-4}, \quad \varphi_{2m-4} = \bar{F}_{2m-4}, \quad u_{2m-6} = \frac{2}{3}u_{2m-8} + \\
&\frac{1}{3}w_2 + \bar{F}_{2m-6}, \quad \bar{F}_{2m-6} = \frac{4}{3}\left(\frac{1}{2}\bar{F}_{2m-4} + \varphi_{2m-6}\right), \cdots, \\
&u_{2m-2k-2} = \frac{k}{k+1}u_{2m-2k-4} + \frac{1}{k+1}w_2 + \bar{F}_{2m-2k-2}, \\
&\bar{F}_{2m-2k-2} = \frac{2k}{k+1}\left(\frac{1}{2}\bar{F}_{2m-2k} + \varphi_{2m-2k-2}\right).
\end{aligned} \tag{16}
$$

Taking $k = m-3$ in (16a) and using (9c), we have the following relation:

$$
-\frac{1}{2(m-2)}w_2 - \frac{1}{2}\nu_1 + \frac{3m-4}{4(m-2)}\nu_2 = \bar{\phi}_2, \quad \bar{\phi}_2 = \frac{1}{2}\bar{F}_4 + \varphi_2. \tag{17}
$$

5. Finally, from the equations (11a), (13a), (15a) and (17a) we get the following reduced (4×4) system for the parameters ν_α, w_α:

$$
\begin{bmatrix} a & 0 & b & c \\ 0 & d & e & b \\ b & e & d & 0 \\ c & b & 0 & a \end{bmatrix}
\begin{bmatrix} \nu_1 \\ \nu_2 \\ w_2 \\ w_1 \end{bmatrix}
=
\begin{bmatrix} \phi_1 \\ \phi_2 \\ \bar{\phi}_2 \\ \bar{\phi}_1 \end{bmatrix} \tag{18}
$$

where $a = -\dfrac{1}{2(m-1)}$, $b = -\dfrac{1}{2}$, $c = \dfrac{4m-3}{2(m-1)}$, $d = -\dfrac{1}{2(m-2)}$, $e = \dfrac{3m-4}{4(m-2)}$

Note that the parameters $v_\alpha v_a, w_a$ in (18) is ordered in such a way that the coefficient matrix is to be symmetric with respect to both diagonals. The determinant of this system $\Delta = \dfrac{5m(5m-4)}{16(m-1)(m-2)} \neq 0$.

6. The solution of the system (18) is given below in a successive order:

$$
\begin{aligned}
&w_1 = \frac{\beta\rho_2 - \alpha\rho_1}{\beta\gamma - \alpha^2}, \qquad w_2 = \frac{1}{\alpha}(\rho_2 - \gamma w_1), \\
&\nu_1 = \frac{1}{\alpha}(\phi_1 - bw_2 - cw_1), \qquad \nu_2 = \frac{1}{d}(\phi_2 - ew_2 - bw_1),
\end{aligned} \tag{19}
$$

where, $\alpha = b(cd + ae)$, $\beta = -ad^2 + ae^2 + db^2$, $\gamma = -ad^2 + ab^2 + dc^2$, $\rho_1 = -ad\bar{\phi}_2 + bd\phi_1 + ae\phi_2$, $\rho_2 = -ad\bar{\phi}_1 + cd\phi_1 + ab\phi_2$.

7. To find the remaining values $u_i (i = \overline{3, 2m-3})$, two point recurrence relations given in either (10) and (12) or in (14) and (16) can be used.

8. It is right the following theorem on the stability of the above recurrence relations:

Theorem-1. Let the right parts of the recurrence relations have the following forms:

$$F_{2k+1} = \frac{2k}{k+1}\left[\frac{1}{2}F_{2k-1} + \varphi_{2k+1}\right], \quad (k = \overline{2, m-2})$$

or

$$F_{2k+2} = \frac{2k}{k+1}\left[\frac{1}{2}F_{2k} + \varphi_{2k+2}\right], \quad (k = \overline{2, m-3}),\ n = 2m$$

Then $|F_i| \le \frac{1}{2m}M,\ M = \max_i |-4f(x_i)|$.

These estimates are also right for $|\bar{F}_i|,\ (i = \overline{3, 2m-3})$.

9. The numerical results of the solution of BVP-1 is shown comparatively in Table-1 for $n = 32$. The results of direct application of finite difference method (with 3-point central difference stencil approximation for the second order derivative) is shown as $DIRECT_{FD}$ and the results of combination of variational and numerical methods named as VASHA. Both results are compared with the exact result and percent relative errors are also tabulated. Some columns in the table filled with blue indicating that at those points the combined method gives the better results.

Table-1 Numerical Results for the BVP-1

Results for n=32	MIN % ERROR		MAX % ERROR		**u1**	**u2**	**u3**	**u4**	**u5**	**u6**	**u7**	**u8**
% error VASHA					16.09	13.95	0.92	2.34	1.44	0.47	1.37	0.06
u_VASHA	0.06	u8	16.09	u1	0.00076904	0.0029541	0.0072845	0.01168	0.01763	0.02310	0.02961	0.03518
u_EXACT					**0.00091648**	**0.0034332**	**0.0072184**	**0.01196**	**0.01738**	**0.02321**	**0.02921**	**0.03516**
u_DIRECT_FD	0.38	u16	3.23	u1	0.00094604	0.0034904	0.0073013	0.01207	0.01751	0.02336	0.02937	0.03534
% error D_FD					3.23	1.67	1.15	0.92	0.75	0.65	0.55	0.51

Results for n=32	**u9**	**u10**	**u11**	**u12**	**u13**	**u14**	**u15**	**u16**	**BVP-1**
% error VASHA	1.30	0.28	1.20	0.38	1.17	0.43	1.14	0.43	$u'' = 12x^2 - 12x + 2$,
u_VASHA	0.04139	0.04629	0.05150	0.05514	0.05886	0.06082	0.06272	0.06277	$u\|_{\partial\Omega} = 0, x \in \Omega = (0,1)$.
u_EXACT	**0.04086**	**0.04616**	**0.05089**	**0.05493**	**0.05818**	**0.06056**	**0.06201**	**0.06250**	**SOLN-1** $u_{exact} = x^2(1-x)^2$
u_DIRECT_FD	0.04106	0.04637	0.05111	0.05516	0.05842	0.06080	0.06226	0.06274	Note that the remaining values are
% error D_FD	0.49	0.45	0.43	0.42	0.41	0.40	0.40	0.38	symmetric with respect to value of u16.

It should be notted that combined application of variational and numerical mehods reduces the order of derivative to be approximated to half. By this way, with this combined method, relatively less smooth boundary value problems like BVP-2 given below can be solved and generalised solutions can also be obtained.

4 BVP-2: Example for Less Smooth BVP

Consider the following BVP

$$u'' = \begin{cases} 0, & x \in \left[0, \frac{1}{3}\right] \\ 1, & x \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ 0, & x \in \left[\frac{2}{3}, 1\right]. \end{cases} \quad (20)$$

$$u|_{\partial\Omega} = 0, \ \ u := u(x) \in C^1(\Omega) \cap C(\bar{\Omega}), \ \ x \in \Omega := (0, 1). \quad (21)$$

5 Solution of the BVP-2

The exact solution of the BVP given in (20) is

$$u'' = \begin{cases} -\frac{1}{6}x, & x \in \left[0, \frac{1}{3}\right]; \\ \frac{1}{2}\left(x^2 - x + \frac{1}{9}\right), & x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\ \frac{1}{6}(x-1), & x \in \left[\frac{2}{3}, 1\right]. \end{cases} \quad (22)$$

The numerical solution of the BVP-2 is given both in numbers (see Table-2) and in graphical representation (see Figure-1). The result of this paper shows a regular fluctuating behavior with respect to both the finite difference and the exact results. As moving from two adjacent points one of the values approximates from the upper side and the other value from the lower side. This fluctuating behaviour gives us a possibility of offering better approximations. For example, it is clear from the figure and can be checked from the table that the arithmetical average of the successive function values almost coincides with the corresponding finite difference results. Further investigations are needed to develop new strategies to have better approximations.

Table-2 Numerical Results for the BVP-2

Figure-1 Graphical Representation of the Numerical Results for the BVP-2

Results for n=32	**MIN % ERROR**		**MAX % ERROR**		**u1**	**u2**	**u3**	**u4**	**u5**	**u6**	**u7**	**u8**
% error VASHA					5.00	2.50	9.98	4.37	11.02	4.99	11.44	5.33
u_VASHA	2.50	u2	11.91	u11	-0.0054688	-0.01016	-0.01719	-0.01992	-0.02891	-0.02969	-0.04063	-0.03945
u_EXACT					**-0.0052083**	**-0.01042**	**-0.01563**	**-0.02083**	**-0.02604**	**-0.03125**	**-0.03646**	**-0.04167**
u_DIRECT_FD	2.66	u16	3.21	u11	-0.0053711	-0.01074	-0.01611	-0.02148	-0.02686	-0.03223	-0.03760	-0.04297
% error D_FD					3.12	3.07	3.07	3.12	3.15	3.14	3.13	3.12

Results for n=32	**u9**	**u10**	**u11**	**u12**	**u13**	**u14**	**u15**	**u16**
% error VASHA	11.65	5.49	11.91	4.30	10.48	3.93	9.89	3.80
u_VASHA	-0.05234	-0.04922	-0.06406	-0.05898	-0.07187	-0.06484	-0.07578	-0.06680
u_EXACT	**-0.04688**	**-0.05208**	**-0.05724**	**-0.06163**	**-0.06505**	**-0.06749**	**-0.06896**	**-0.06944**
u_DIRECT_FD	-0.04834	-0.05371	-0.05908	-0.06348	-0.06689	-0.06934	-0.07080	-0.07129
% error D_FD	3.11	3.13	3.21	3.00	2.83	2.74	2.67	2.66

BVP-2

$$u'' = \begin{cases} 0, x \in \left[0, \frac{1}{3}\right]; \\ 1, x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\ 0, x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

SOLN-2

$$u_e = \begin{cases} \frac{-1}{6} x, & x \in \left[0, \frac{1}{3}\right]; \\ \frac{1}{2}\left(x^2 - x + \frac{1}{9}\right), & x \in \left(\frac{1}{3}, \frac{2}{3}\right); \\ \frac{1}{6}(x-1), & x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

$u|_{\partial\Omega} = 0, u := u(x) \in C^1(\Omega) \cap C(\bar{\Omega}), x \in \Omega := (0,1).$

Note that the remaining values are symmetric with respect to value of u16.

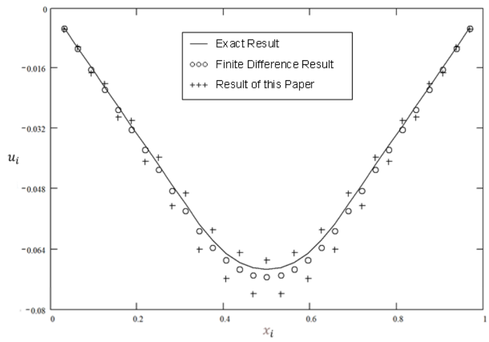

Figure 1: Graphical Representation of the Numerical Results for the BVP-2

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