SOLUTION OF THE SOME BOUNDARY VALUE PROBLEM FOR ELASTIC MATERIALS WITH VOIDS IN THE CASE OF APPROXIMATION N=1 OF VEKUA'S

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Abstract

In this paper we consider a boundary value problem for a circle. The plate is the elastic material with voids. The state of plate equilibrium is described by the system of differential equations that is derived from three dimensional equations of equilibrium of an elastic material with voids (Cowin-Nunziato model) by Vekua's reduction method. its general solution is represented by means of analytic functions of a complex variable and solutions of Helmholtz equations. The problem is solved analytically by the method of the theory of functions of a complex variable.

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1 Introduction

Using the concept of volume fraction of pores the theory of elastic materials with single voids is proposed by Nunziato and Cowin [1, 2]. The basic equations of this theory involve the displacement vector field and the change of volume fraction of pores. Such materials include, in particular, rocks and soils, granulated and some other manufactured porous materials.

As is known, there exist many methods of reducing three-dimensional problems of equilibrium of elastic shells to two-dimensional problems. Some such general methods were proposed by famous mathematician and mechanician I. Vekua [3, 4].

Let $Ox_1x_2x_3$ be the rectangular Cartesian coordinate system. Let $\Omega = \omega\times]-h,\ h[$ be an infinite plate with a circular hole of radius R centred at the origin O. The plate thickness is assumed to be constant and equal to $2h$. The plate is the isotropic material with voids.

The governing equations of the theory of elastic materials with voids can be expressed in the following form [2]:

• Equations of equilibrium

$$T_{ij,j} + \Phi_i = 0, \quad j = 1, 2, 3, \tag{1}$$

$$h_{i,i} + g + \Psi = 0, \tag{2}$$

where T_{ij} is the symmetric stress tensor, Φ_i are the volume force components, h_i is the equilibrated stress vector, g is the intrinsic equilibrated body force and Ψ is the extrinsic equilibrated body force.

• Constitutive equations

$$\begin{array}{l} T_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \beta\phi\delta_{ij}, \quad i,j = 1,2,3, \\ h_i = \alpha\phi_{,i}, \quad i = 1,2,3, \quad g = -\xi\phi - \beta e_{kk}, \end{array} \tag{3}$$

where λ and μ are the Lamé constants; α, β and ξ are the constants characterizing the body porosity; δ_{ij} is the Kronecker delta; $\phi := \nu - \nu_0$ is the change of the volume fraction function from the matrix reference volume fraction ν_0 (clearly, the bulk density $\rho = \nu\gamma$, $0 < \nu \leq 1$, here γ is the matrix density and ρ is the mass density); e_{ij} is the strain tensor and

$$e_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{4}$$

where $u_i,\ i = 1, 2, 3$ are the components of the displacement vector..

The constitutive equations also meet some other conditions, following from physical considerations

$$\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad 3\lambda + 2\mu > 0, \qquad (3\lambda + 2\mu)\xi > 3\beta^2. \tag{5}$$

2 Basic equations

Using Vekua's dimension reduction method [3], linear two-dimensional (2D) governing equations were obtained from the above three-dimensional (3D) equations with respect to so-called r-th order moments of functions under consideration, where the zero order moments (which are averaged along the thickness of the plate) and the first order moments are defined as

$$\left(\overset{(0)}{T}_{ij}, \overset{(0)}{h}_i, \overset{(0)}{g}, \overset{(0)}{u}_i, \overset{(0)}{\phi}\right) = \frac{1}{2h}\int\limits_{-h}^{h} \left(T_{ij}, h_i, g, u_i, \phi\right) dx_3,$$

$$\left(\overset{(1)}{T}_{ij}, \overset{(1)}{h}_i, \overset{(1)}{g}, \overset{(1)}{u}_i, \overset{(1)}{\phi}\right) = \frac{3}{2h^2}\int_{-h}^{h} x_3 \cdot (T_{ij}, h_i, g, u_i, \phi)\, dx_3.$$

In particular, in the $N = 1$ approximation of I.Vekua's theory it is assumed that

$$(u_i,\ \phi)(x_1, x_2, x_3) = \left(\overset{(0)}{u}_i,\ \overset{(0)}{\phi}\right)(x_1, x_2) + \frac{x_3}{h}\left(\overset{(1)}{u}_i,\ \overset{(1)}{\phi}\right)(x_1, x_2).$$

For $h = const$ the reduced system of equilibrium equations gets split into two independent systems: tension−compression equations with unknowns $\overset{(0)}{u}_1$, $\overset{(0)}{u}_2$, $\overset{(1)}{u}_3$, $\overset{(0)}{\phi}$ and bending equations with unknowns $\overset{(1)}{u}_1$, $\overset{(1)}{u}_2$, $\overset{(0)}{u}_3$, $\overset{(1)}{\phi}$. In this paper we consider the system of tension−compression equations.

from [5] the basic relations of the $N = 1$ approximation of elastic isotropic plates with voids have the following form:

$$\partial_\alpha \overset{(0)}{T}_{\alpha\gamma} = 0,\ \ \alpha, \gamma = 1, 2\ \ \partial_\alpha \overset{(1)}{T}_{\alpha 3} - \frac{3}{h}\overset{(1)}{T}_{33} = 0,\ \ \partial_\alpha \overset{(0)}{h}_\alpha + \overset{(0)}{g} = 0, \tag{6}$$

where

$$\begin{aligned}
&\overset{(0)}{T}_{\alpha\gamma} = \lambda\left(\overset{(0)}{\theta} + \overset{(1)}{u}_3\right)\delta_{\alpha\gamma} + \mu\left(\partial_\alpha \overset{(0)}{u}_\gamma + \partial_\gamma \overset{(0)}{u}_\alpha\right) + \beta\, \overset{(0)}{\phi}\, \delta_{\alpha\gamma},\\
&\overset{(0)}{T}_{33} = \lambda\left(\overset{(0)}{\theta} + \overset{(1)}{u}_3\right) + 2\mu \overset{(1)}{u}_3 + \beta\, \overset{(0)}{\phi},\ \ \overset{(1)}{T}_{\gamma 3} = \mu\partial_\gamma \overset{(1)}{u}_3,\ \ \overset{(0)}{h}_\gamma = \alpha\partial_\gamma \overset{(0)}{\phi},\\
&\overset{(0)}{g} = -\xi\, \overset{(0)}{\phi} - \beta\left(\overset{(0)}{\theta} + \overset{(1)}{u}_3\right),\ \ \overset{(0)}{\theta} = \partial_1 \overset{(0)}{u}_1 + \partial_2 \overset{(0)}{u}_2.
\end{aligned} \tag{7}$$

On the plane Ox_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar z} = 0.5(\partial_1 + i\partial_2)$, $\bar z = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar z}$.

Substituting (7) into system (6), we obtain the following system of governing equations of statics with respect to the functions $\overset{(0)}{u}_1$, $\overset{(0)}{u}_2$, $\overset{(1)}{u}_3$, $\overset{(0)}{\phi}$ in the complex form

$$\begin{aligned}
&2\mu\partial_{\bar z}\partial_z \overset{(0)}{u}_+ + (\lambda+\mu)\partial_{\bar z}\overset{(0)}{\theta} + \lambda\partial_{\bar z}\overset{(1)}{u}_3 + \beta\partial_{\bar z}\overset{(0)}{\phi} = 0,\\
&\mu\Delta \overset{(1)}{u}_3 - \frac{3}{h}\left[\lambda\overset{(0)}{\theta} + (\lambda+2\mu)\overset{(1)}{u}_3 + \beta\overset{(0)}{\phi}\right] = 0,\\
&(\alpha\Delta - \xi)\overset{(0)}{\phi} - \beta\left[\overset{(0)}{\theta} + \overset{(1)}{u}_3\right] = 0,
\end{aligned} \tag{8}$$

where $\overset{(0)}{u}_+ = \overset{(0)}{u}_1 + i\,\overset{(0)}{u}_2$, $\overset{(0)}{\theta} = \partial_z \overset{(0)}{u}_+ + \partial_{\bar z}\overset{(0)}{\bar u}_+$.

As the analogues of the Kolosov-Muskhelishvili formulas [6] for system (8) we have

$$\begin{aligned}
&2\mu\overset{(0)}{u}_{+} = \varkappa_1\varphi(z) - \varkappa_2 z\overline{\varphi'(z)} - \overline{\psi(z)} - p_1\partial_{\bar z}\chi_1(z,\bar z) - p_2\partial_{\bar z}\chi_2(z,\bar z),\\
&\overset{(1)}{u}_3 = l_{11}\chi_1(z,\bar z) + l_{12}\chi_2(z,\bar z) - E_1(\varphi'(z) + \overline{\varphi'(z)}),\\
&\overset{(0)}{\phi} = l_{21}\chi_1(z,\bar z) + l_{22}\chi_2(z,\bar z) - E_2(\varphi'(z) + \overline{\varphi'(z)}),
\end{aligned} \tag{9}$$

where $\varphi(z)$ and $\psi(z)$ are the arbitrary analytic functions of z, $\chi_1(z,\bar z)$ and $\chi_2(z,\bar z)$ are the general solutions of the Helmholtz equations

$$\Delta\chi - \kappa_1\chi = 0, \quad \Delta\chi - \kappa_2\chi = 0,$$

and κ_1, κ_2 are eigenvalues and $l_{11}, l_{21}, l_{12}, l_{22}$ are eigenvectors of the matrix C. $E_1 = a_{11} + a_{12}$, $E_2 = a_{21} + a_{22}$ and a_{ij} are coefficients of the matrix $-C^1D$:

$$C = \begin{pmatrix} \frac{12(\lambda+\mu)}{h(\lambda+2\mu)} & \frac{6\beta}{h(\lambda+2\mu)} \\ \frac{2\mu\beta}{\alpha(\lambda+2\mu)} & \frac{\xi}{\alpha} - \frac{\beta^2}{\alpha(\lambda+2\mu)} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{3\lambda}{2h\mu(\lambda+2\mu)} & 0 \\ 0 & \frac{\beta}{2\alpha(\lambda+2\mu)} \end{pmatrix}.$$

Also $\varkappa_1 = \frac{1}{2} + \frac{(\lambda E_1+\beta E_2)\mu}{\lambda+2\mu}$, $\varkappa_2 = \frac{1}{2} - \frac{(\lambda E_1+\beta E_2)\mu}{\lambda+2\mu}$, $p_1 = \frac{4(\lambda l_{11}+\beta l_{21})\mu}{\kappa_1(\lambda+2\mu)}$, $p_2 = \frac{4(\lambda l_{12}+\beta l_{22})\mu}{\kappa_2(\lambda+2\mu)}$.

Complex combinations of the stress tensor components are expressed by means of the formulas

$$\overset{(0)}{T}_{11} - \overset{(0)}{T}_{22} + 2i\overset{(0)}{T}_{12} = -2\varkappa_2 z\overline{\varphi''(z)} - \overline{\psi'(z)} - 2p_1\partial^2_{\bar z\bar z}\chi_1(z,\bar z) - 2p_2\partial^2_{\bar z\bar z}\chi_2(z,\bar z),$$

$$\begin{aligned}
&\overset{(0)}{T}_{11} + \overset{(0)}{T}_{22} = E_3(\varphi'(z) + \overline{\varphi'(z)}) + E_4\chi_1(z,\bar z) + E_5\chi_2(z,\bar z),\\
&\overset{(1)}{T}_{+} = l_{11}\partial_{\bar z}\chi_1(z,\bar z) + l_{12}\partial_{\bar z}\chi_2(z,\bar z) - E_1\overline{\varphi''(z)}),\\
&\overset{(0)}{h}_{+} = l_{21}\partial_{\bar z}\chi_1(z,\bar z) + l_{22}\partial_{\bar z}\chi_2(z,\bar z) - E_2\overline{\varphi''(z)}),
\end{aligned} \tag{10}$$

where

$$E_3 = \frac{\lambda+\mu}{\mu}(\varkappa_1 + \varkappa_2) - 2\lambda E_1 - 2\beta E_2, \quad E_4 = 2\lambda l_{11} + 2\beta l_{21} - \frac{\lambda+\mu}{\mu}8p_1\kappa_1,$$

$$E_5 = 2\lambda l_{12} + 2\beta l_{22} - \frac{\lambda+\mu}{\mu}8p_2\kappa_2.$$

3 The boundary value problem for a circle

Let us consider the elastic circle with voids bounded by the circumference of radius R (Fig. 1). The origin of coordinates is at the center of the circle.

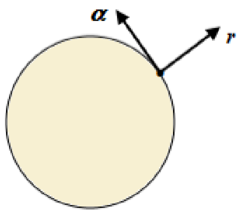

Figure 1: The elastic circle.

On the circumference, we consider the following boundary value problem

$$\overset{(0)}{T}_{rr} + i\overset{(0)}{T}_{r\vartheta} = N - iT, \quad \overset{(1)}{T}_{r3} = M, \quad \overset{(0)}{h}_{r3} = F. \tag{11}$$

The boundary conditions take the form

$$\begin{aligned}
&\overset{(0)}{T}_{rr} + i\overset{(0)}{T}_{r\vartheta} = E_3(\varphi'(z) + \overline{\varphi'(z)}) + E_4\chi_1(z,\bar z) + E_5\chi_2(z,\bar z)\\
&-\left(2\varkappa_2 z\overline{\varphi''(z)} + \overline{\psi'(z)} + 2p_1\partial^2_{\bar z\bar z}\chi_1(z,\bar z) + 2p_2\partial^2_{\bar z\bar z}\chi_2(z,\bar z)\right)e^{-2i\vartheta}\\
&\overset{(1)}{T}_{r3} = \left(l_{11}\partial_{\bar z}\chi_1(z,\bar z) + l_{12}\partial_{\bar z}\chi_2(z,\bar z) - E_1\overline{\varphi''(z)}\right)e^{-i\vartheta}\\
&+\left(l_{11}\partial_z\chi_1(z,\bar z) + l_{12}\partial_z\chi_2(z,\bar z) - E_1\varphi''(z)\right)e^{i\vartheta},\\
&\overset{(1)}{h}_{r3} = \left(l_{21}\partial_{\bar z}\chi_1(z,\bar z) + l_{22}\partial_{\bar z}\chi_2(z,\bar z) - E_2\overline{\varphi''(z)}\right)e^{-i\vartheta}\\
&+\left(l_{21}\partial_z\chi_1(z,\bar z) + l_{22}\partial_z\chi_2(z,\bar z) - E_2\varphi''(z)\right)e^{-i\vartheta}.
\end{aligned} \tag{12}$$

The analytic functions $\varphi'(z)$, $\psi'(z)$ and the metaharmonic functions $\chi_1(z,\bar z)$ and $\chi_2(z,\bar z)$ are represented as the series

$$\begin{gathered}
\varphi'(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \psi'(z) = \sum_{n=0}^{\infty} b_n z^n,\\
\chi_1(z,\bar z) = \sum_{-\infty}^{+\infty} \alpha_n I_n(\sqrt{\kappa_1}r)e^{in\vartheta}, \quad \chi_2(z,\bar z) = \sum_{-\infty}^{+\infty} \beta_n I_n(\sqrt{\kappa_2}r)e^{in\vartheta}
\end{gathered} \tag{13}$$

where $I_n(\cdot)$ are the modified Bessel function of the first kind of n-th order. From (5) κ_1 and κ_2 are positive numbers.

Expand the function $N - iT$, M and F, given on $r = R$, in a complex Fourier series

$$N - iT = \sum_{-\infty}^{+\infty} A_n e^{in\alpha}, \quad M = \sum_{-\infty}^{+\infty} B_n e^{in\alpha}, \quad F = \sum_{-\infty}^{+\infty} C_n e^{in\alpha}. \tag{14}$$

Substituting (12), (13), (14) into (11) and comparing the coefficients of same exponents we obtain

$$\begin{aligned}
&2E_3a_0 + \left(E_4I_0(\sqrt{\kappa_1}R) - \frac{p_1\kappa_1}{2}I_2(\sqrt{\kappa_1}R)\right)\alpha_0\\
&+\left(E_5I_0(\sqrt{\kappa_2}R) - \tfrac{p_2\kappa_2}{2}I_2(\sqrt{\kappa_2}R)\right)\beta_0 = A_0,
\end{aligned}$$

$$\begin{aligned} &l_{11}\sqrt{\kappa_1}I_1(\sqrt{\kappa_1}R)\alpha_0 + l_{12}\sqrt{\kappa_2}I_1(\sqrt{\kappa_2}R)\beta_0 = B_0, \\ &l_{21}\sqrt{\kappa_1}I_1(\sqrt{\kappa_1}R)\alpha_0 + l_{22}\sqrt{\kappa_2}I_1(\sqrt{\kappa_2}R)\beta_0 = C_0, \end{aligned} \tag{15}$$

$$\begin{aligned} &E_3R^na_n + \left(E_4I_n(\sqrt{\kappa_1}R) - \frac{p_1\kappa_1}{2}I_{n+2}(\sqrt{\kappa_1}R)\right)\alpha_n \\ &+ \left(E_5I_n(\sqrt{\kappa_2}R) - \tfrac{p_2\kappa_2}{2}I_{n+2}(\sqrt{\kappa_2}R)\right)\beta_n = A_n,\ n > 0 \\ &(E_3 - 2\varkappa_2 n)R^na_n + \left(E_4I_n(\sqrt{\kappa_1}R) - \frac{p_1\kappa_1}{2}I_{n-2}(\sqrt{\kappa_1}R)\right)\alpha_n \\ &+ \left(E_5I_n(\sqrt{\kappa_2}R) - \tfrac{p_2\kappa_2}{2}I_{n-2}(\sqrt{\kappa_2}R)\right)\beta_n - R^{n-2}b_{n-2} = \overline{A}_{-n}, n > 0 \\ &-E_1nR^{n-1}a_n + \frac{l_{11}\sqrt{\kappa_1}}{2}(I_{n-1}(\sqrt{\kappa_1}R) + I_{n+1}(\sqrt{\kappa_1}R))\alpha_n \\ &+\tfrac{l_{12}\sqrt{\kappa_2}}{2}(I_{n-1}(\sqrt{\kappa_2}R) + I_{n+1}(\sqrt{\kappa_2}R))\beta_n = B_n,\ n > 0 \\ &-E_2nR^{n-1}a_n + \frac{l_{21}\sqrt{\kappa_1}}{2}(I_{n-1}(\sqrt{\kappa_1}R) + I_{n+1}(\sqrt{\kappa_1}R))\alpha_n \\ &+\tfrac{l_{22}\sqrt{\kappa_2}}{2}(I_{n-1}(\sqrt{\kappa_2}R) + I_{n+1}(\sqrt{\kappa_2}R))\beta_n = C_n,\ n > 0. \end{aligned} \tag{16}$$

All coefficients in series (13) are found by solving (15)-(16). It is easy to prove the absolute and uniform convergence of the series obtained in the circular ring (including the contours) when the functions set on the boundaries have sufficient smoothness.

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