GALERKIN'S METHOD AS CORRECTED BY BASTATSKY AND KHVOLES

Isaac Elishakoff

Department of Ocean and Mechanical Engineering,
Florida Atlantic University,
Boca Raton, FL 33431- 0991, USA
elishako@fau.edu

Abstract

This paper is devoted to the tale of correction of the Galerkin method as facilitated by Bastatsky and Khvoles nearly half a century ago, in 1972 and expanded recently by the present writer and his collaborators. It is dedicated to the centenary of Alexander Rubenovich Khvoles-a great scientist and man.

1 Introduction

Galerkin method is a widely used method in mathematical physics suggested in 1915 by Galerkin. The books by Michlin (1971) and Svirsky (1968), for example, can be consulted with its description and convergence. The centenary of the method and its achievements are discussed by Repin (2017). Singer (1962) and others showed that it coincides with the Ritz method if the utilized coordinate functions satisfy all boundary conditions. Timoshenko (1953) failed to mention this method among the important contributions to mechanics, in his history book. Moreover, in his textbook Timoshenko unjustifiably maintained that the Galerkin method constitutes a second form of the Ritz Method. It was shown by Elishakoff, Kaplunov, and Kaplunov (2020), and by Reddy and Srinivasa (2020) that the Ritz and Galerkin methods are two distinct methods.

Two scientists working in Georgia, Bastatsky and Alexander Rubenovich Khvoles (1972) a problem of application of the Galerkin method to stepped structures. They showed that the nave application of Galerkin method does not yield results produced by the Ritz method. This writer was a new arrival to Israel in general and in Technion-Israel Institute of technology in particular. The Department of Aeronautical engineering was getting some journals in Russian, like Mechanics of Solids, and Mechanics Referee Journal. From the latter I found that the journal Structural Mechanics and Analysis of Constructions, published the paper on the Galerkin method. Being interested in this method, I went to the faculty of Civil Engineering, whose library was receiving this journal.

Present writer was impressed both by the beautiful idea and clear presentation of application of generalized functions, namely the Heavisde unit function, Dirac's delta function, and the doublet function, namely the derivative of the Dirac's delta function. I did not have a chance to return to these problems until recently, publishing several sequels.

Present writer contacted Alexander Rubenovich asking if he had any further ideas on the Galerkin method's extension or on applicability of the methodology that he had suggested with Bastatsky. He responded that at that time he didn't have. Here is his response, dated August 16, 2002, obtained via Anna Afonchenko: Present writer tried to work on this

Глубокоуважаемый Исаак Элишаков!

Ваше письмо от 26.07.02. и копию работы «О некоторых особенностях ...» я получил – спасибо. Другая работа с Бастацким «К вопросу о применении метода Б.-Г. ...», в сборнике «Сопротивление материалов и теория сооружений», Киев, 1972, вып. 6, стр. 76-78. Ни первой, ни второй работы я у себя не обнаружил. (Я нашёл в списке моих работ.) Так как эти работы опубликованы в 1972 г., то прошло 30 лет. Возможно, теперь я подошёл бы иначе. Если появится идея, обязательно Вам сообщу. Прошу сообщить Ваше отчество.

Всего самого хорошего.

Ваш А.Р.Хволес

method, thinking to get it done by his centenary celebration. Whereas this event is taking place now, by the generosity of spirit of his Georgian colleagues, he is not with us anymore, physically. It is with delight that I dedicate this account to his blessed memory.

2 Basic Equations

We are interested to evaluate the natural frequencies of a multi-step beam as shown in Fig.1.

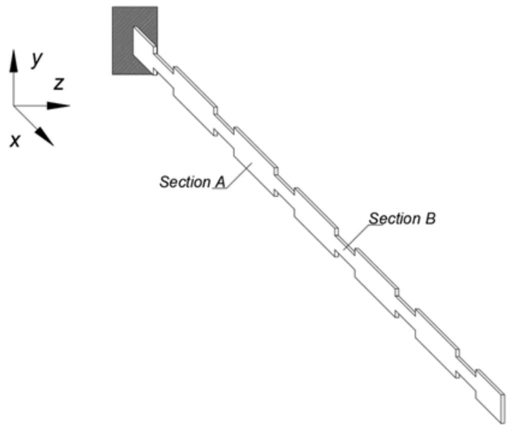
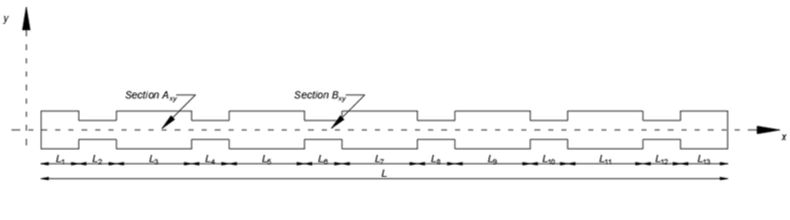


Figure 1: Schematic of 13-stepped beam of length L

The beam is a cantilever made of a single material so the elastic modulus E and the mass density ρ are constants. The beam is composed by two alternating sections, namely section A and section B. We study the free vibrations of this beam in both vertical $x-y$ and horizontal $x-z$ planes as shown in Fig. 2.



(a)

Figure 2: of the stepped beam in the plane $x-y$ (a) and $x-z$ (b)

The Euler-Bernoulli differential equation governing the flexural vibrations in one principal plane of the non-uniform beam reads:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 w}{\partial x^2}\right)+\rho(x)A(x)\frac{\partial^2 w}{\partial t^2}=0, \tag{1}$$

where $w(x,t)$ is the vertical displacement, $I(x)$ the moment of inertia, $A(x)$ the cross-sectional area, x the axial coordinate and t is the time. For each segment of the stepped beam, one can write:

$$E_jI_j\frac{\partial^4 w}{\partial x^4}+\rho_jA_j\frac{\partial^2 w}{\partial t^2}=0, \tag{2}$$

where j is an integer which identifies the segment of the beam. We rewrite the vertical displacement as follows:

$$w(x,t) = W(x)\sin(\omega t), \tag{3}$$

where ω is the sought natural frequency of the beam. Substituting Eq. (3) in the differential equations (2) we easily obtain the following set of equations valid for any time instant:

$$\frac{d^4W}{dx^4} - \alpha_j^4 W = 0, \tag{4}$$

where α_j reads:

$$\alpha_j = \sqrt[4]{\frac{\rho_j A_j \omega^2}{E_j I_j}}. \tag{5}$$

As well known, the mode shapes $W_j(x)$ are given by:

$$W_j(x) = D_{1,j}\sin(\alpha_j x) + D_{2,j}\cos(\alpha_j x) + D_{3,j}\cosh(\alpha_j x) + D_{4,j}\sinh(\alpha_j x). \tag{6}$$

These satisfy the differential equations in Eq. (4), where $D_{i,j}$ are constants of integration. We now take advantage of the Krylov-Duncan functions to rewrite Eq. (6). The Krylov-Duncan functions are four functions defined as follows:

$$K_1(\alpha x) = \frac{1}{2}[\cosh(\alpha x) + \cos(\alpha x)], \tag{7a}$$

$$K_2(\alpha x) = \frac{1}{2}[\sinh(\alpha x) + \sin(\alpha x)], \tag{7b}$$

$$K_3(\alpha x) = \frac{1}{2}[\cosh(\alpha x) - \cos(\alpha x)], \tag{7c}$$

$$K_4(\alpha x) = \frac{1}{2}[\sinh(\alpha x) - \sin(\alpha x)]. \tag{7d}$$

One notes that:

$$K_1(0) = 1, \tag{8a}$$

$$K_2(0) = 0, \tag{8b}$$

$$K_3(0) = 0, \tag{8c}$$

$$K_4(0) = 0. \tag{8d}$$

The second property of these functions is that the first derivative of K_i is equal to K_{i-1}:

Krylov-Duncan Function	$K_1(x)$	$K_2(x)$	$K_3(x)$	$K_4(x)$
First derivative	$K_4(x)$	$K_1(x)$	$K_2(x)$	$K_3(x)$
Second derivative	$K_3(x)$	$K_4(x)$	$K_1(x)$	$K_2(x)$
Third derivative	$K_2(x)$	$K_3(x)$	$K_4(x)$	$K_1(x)$

Table 1: Derivatives of Krylov-Duncan functions

We can use these functions into equation (6) in order to simplify the representation of the boundary conditions. This will lead us to the following equation:

$$W_j(x) = M_{1,j}K_1(\alpha_j x) + M_{2,j}K_2(\alpha_j x) + M_{3,j}K_3(\alpha_j x) + M_{4,j}K_4(\alpha_j x), \quad (9)$$

where $M_{i,j}$ are constant of integration.

3 Exact Solution

The evaluation of the exact solution consists in the demand that not all four coefficients $M_{i,j}$ for each component vanish simultaneously. In our study we have 13 different segments for the multi-step beam resulting in 52 unknowns. The solution should satisfy continuity conditions between the segments and the boundary conditions at the outer sections of the beams (first and the 13^{th} components). For each discontinuity, we have four compatibility conditions namely continuity of vertical displacement, slope, bending moment and shear force, for a total of 48 equations of compatibility given the 12 discontinuities in the beam. In particular, they read:

$$W_j(x = L_j) = W_{j+1}(x = L_j) \quad (10a),$$

$$\frac{dW_j}{dx}(x = L_j) = \frac{dW_{j+1}}{dx}(x = L_j) \quad (10b),$$

$$E_jI_j\frac{d^2W_j}{dx^2}(x = L_j) = E_{j+1}I_{j+1}\frac{d^2W_{j+1}}{dx^2}(x = L_j), \quad (10c)$$

$$E_jI_j\frac{d^3W_j}{dx^3}(x = L_j) = E_{j+1}I_{j+1}\frac{d^3W_{j+1}}{dx^3}(x = L_j). \quad (10d)$$

By adding the 4 boundary conditions at the extremes of the beam we can formulate a problem with 52 equations for 52 unknowns. In particular, in the following, to compare our results with those of Jarowski and Dowell (2008) we consider the case of the cantilever beam which boundary conditions read:

Constrain conditions		$x = 0$	$x = L$
	Cantilever	$W_1 = 0$ $\frac{dW_1}{dx} = 0$	$E_{13}I_{13}\frac{d^2W_{13}}{dx^2} = 0$ $E_{13}I_{13}\frac{d^3W_{13}}{dx^3} = 0$

This system of equations has the following form:

$$\tilde{A}\bar{x} = \bar{0}, \tag{11}$$

where $\tilde{A}$ is the coefficient matrix, $\bar{x}$ the vector of unknowns and $\bar{0}$ denotes the zero vector. The non-trivial solutions of the homogeneous system in eq (11) lead to the natural frequencies ω of the problem.

The matrix $\tilde{A}$ is sparse and the non-zero terms appear around the main diagonal.

4 Straightforward Galerkin Method

The Galerkin method is a numerical strategy to solve differential equations in a discrete manner:

$$E_j I_j \frac{d^4W}{dx^4} - \rho_j A_j \omega^2 W = 0, \quad x_{j-1} < x < x_j. \tag{12}$$

By introducing the axial coordinate in non-dimensional form eq (12) can be represented as:

$$E_j I_j \frac{d^4W}{d\xi^4} - \rho_j A_j \omega^2 L^4 W = 0, \quad \xi_{j-1} < \xi < \xi_j. \tag{13}$$

In order to apply the Galerkin method in its straightforward version, we have to express the vertical displacement W in terms of the so-called comparison functions $\psi_p(\xi)$ as:

$$W(\xi) = \sum_{p=1}^{n} a_p \psi_p(\xi), \tag{14}$$

where a_p are unknown constants. Now we substitute the expression of $W(\xi)$ in the differential equations obtaining residuals $\varepsilon_j(\xi)$ since the functions

$\psi_p(\xi)$ do not necessarily satisfy the differential equations:

$$E_j I_j \sum_{p=1}^{n} a_k \frac{d^4\psi_p(\xi)}{d\xi^4} - \rho_j A_j \omega^2 L^4 \sum_{p=1}^{n} a_p \psi_p(\xi) = \varepsilon_j(\xi), \quad \xi_{j-1} < \xi < \xi_j. \tag{15}$$

We now multiply the error $\varepsilon_j(\xi)$ by $\psi_q(\xi)$, we sum it up for all the components and we integrate within j^{th} span:

$$\sum_{p=1}^{n} \left\{ \sum_{j=1}^{13} \left[\int_{\xi_j}^{\xi_{j+1}} E_j I_j \frac{d^4\psi_p(\xi)}{d\xi^4} \psi_q(\xi) d\xi \right] - \omega^2 \sum_{j=1}^{13} \left[\int_{\xi_j}^{\xi_{j+1}} \rho_j A_j L^4 \psi_p(\xi) \psi_q(\xi) d\xi \right] \right\} a_p = 0. \tag{16}$$

By defining:

$$K_{pq} = \sum_{j=1}^{13} \left[\int_{\xi_j}^{\xi_{j+1}} E_j I_j \frac{d^4\psi_p(\xi)}{d\xi^4} \psi_q(\xi) d\xi \right], \tag{17a}$$

$$M_{pq} = \sum_{j=1}^{13} \left[\int_{\xi_j}^{\xi_{j+1}} \rho_j A_j L^4 \psi_p(\xi) \psi_q(\xi) d\xi \right], \tag{17b}$$

we obtain:

$$\sum_{p=1}^{n} (K_{pq} - \omega^2 M_{pq}) a_p = 0. \tag{18}$$

Eq. 18 can be rewritten in matrix notation as:

$$(K - \omega^2 M)a = 0 \tag{19}$$

where K represent the stiffness matrix of the problem, M the mass matrix of the problem and a the vector of the unknown scale factors a_p.

This non-trivial solution of eq. (19) lead to the eigenvalues ω^2 and the scale factors a_p of the problem.

5 Application of Rigorous Galerkin Method by Bastatsky/Khvoles

The rigorous version of the Galerkin method does require generalized functions over the entire domain of the beam length ($0 < x < L$). Starting from Eq. (1) and (3) we obtain:

$$\frac{d^2}{dx^2}\left(E(x)I(x)\frac{d^2W}{dx^2}\right)\sin(\omega t)-\omega^2\rho(x)A(x)W(x)\sin(\omega t)=0. \tag{20}$$

Introducing a non-dimensional axial coordinate ξ and looking for a solution true for any time value, we obtain:

$$\frac{d^2}{d\xi^2}\left(E(\xi)I(\xi)\frac{d^2W}{d\xi^2}\right)-\omega^2L^4\rho(\xi)A(\xi)W(\xi)=0. \tag{21}$$

In order to implement the rigorous Galerkin method we represent the flexural rigidity and the mass of the system as generalized functions:

$$D(\xi)=E(\xi)I(\xi)=E_1I_1\cdot U(\xi)+\sum_{j=1}^{12}[E_{j+1}I_{j+1}-E_jI_j)\cdot H(\xi-\xi_j)], \tag{22a}$$

$$M(\xi)=\rho(\xi)A(\xi)=\rho_1A_1\cdot U(\xi)+\sum_{j=1}^{12}[\rho_{j+1}A_{j+1}-\rho_jA_j)\cdot H(\xi-\xi_j)], \tag{22b}$$

where $H(\xi-\xi_j)$ is the unit step function or Heaviside function which has the following properties:

$$H(\xi-\alpha)=\begin{cases}1 & if\ \xi>\alpha\\ 0 & otherwise\end{cases} \tag{23a}$$

$$\frac{d}{d\xi}H(\xi-\alpha)=\delta(\xi-\alpha), \tag{23b}$$

$$\frac{d}{d\xi}\delta(\xi-\alpha)=\delta'(\xi-\alpha), \tag{23c}$$

where $\delta(\xi)$ is the Dirac's delta function, and $\delta'(\xi-\alpha)$ is the doublet function. Now, rewriting the equation (21) with these considerations we obtain:

$$\frac{d^2}{d\xi^2}\left(D(\xi)\frac{d^2W}{d\xi^2}\right)-\omega^2L^4M(\xi)W(\xi)=0. \tag{24}$$

We evaluate the derivatives to get:

$$D(\xi)\frac{d^4W}{d\xi^4}+2\frac{d}{d\xi}D(\xi)\frac{d^3W}{d\xi^3}+\frac{d^2}{d\xi^2}D(\xi)\frac{d^2W}{d\xi^2}-\omega^2L^4M(\xi)W(\xi)=0. \tag{25}$$

We substitute the approximation in series of $W(\xi)$ (Eq. (14)) arriving at:

$$\begin{aligned}\sum_{p=1}^{n}\Bigg[&D(\xi)\frac{d^4\psi_p(\xi)}{d\xi^4}+2\frac{d}{d\xi}D(\xi)\frac{d^3\psi_p(\xi)}{d\xi^3}\\ &+\frac{d^2}{d\xi^2}D(\xi)\frac{d^2\psi_p(\xi)}{d\xi^2}-\omega^2L^4M(\xi)\psi_p(\xi)\Bigg]a_p=0.\end{aligned} \tag{26}$$

We next multiply the differential equation by $\psi_q(\xi)$ and we integrate it from zero to one, to get:

$$\sum_{p=1}^{n}\Bigg[\int_0^1 D(\xi)\frac{d^4\psi_p(\xi)}{d\xi^4}\psi_q(\xi)d\xi+\int_0^1 2\frac{d}{d\xi}D(\xi)\frac{d^3\psi_p(\xi)}{d\xi^3}\psi_q(\xi)d\xi \\ +\int_0^1\frac{d^2}{d\xi^2}D(\xi)\frac{d^2\psi_p(\xi)}{d\xi^2}\psi_q(\xi)d\xi-\omega^2\int_0^1 L^4M(\xi)\psi_p(\xi)\psi_q(\xi)d\xi\Bigg]a_p \\ =0. \tag{27}$$

By defining:

$$K_{1,pq}=\int_0^1 D(\xi)\frac{d^4\psi_p(\xi)}{d\xi^4}\psi_q(\xi)d(\xi), \tag{28a}$$

$$K_{2,pq}=\int_0^1 2\frac{d}{d\xi}D(\xi)\frac{d^3\psi_p(\xi)}{d\xi^3}\psi_q(\xi)d\xi, \tag{28b}$$

$$K_{3,pq}=\int_0^1 \frac{d^2}{d\xi^2}D(\xi)\frac{d^2\psi_p(\xi)}{d\xi^2}\psi_q(\xi)d\xi, \tag{28c}$$

$$M_{pq}=\int_0^1 L^4M(\xi)\psi_p(\xi)\psi_q(\xi)d\xi \tag{28d}$$

we can rewrite eq. (27) as:

$$\sum_{p=1}^{n}(K_{1,pq}+K_{2,pq}+K_{3,pq}-\omega^2M_{pq})a_p=0 \tag{29}$$

or in more compact matrix form as:

$$(K_1+K_2+K_3-\omega^2M)a=0 \tag{30}$$

Non-trivial solutions of the equation:

$$(K-\omega^2M)a=0 \tag{31}$$

where $K=K_1+K_2+K_3$, lead to the frequencies of vibration ω^2 and the scale factors a_p of the problem.

We observe that the matrix K_1 coincides with the K matrix for the straightforward implementation of the method. Thus, the rigorous implementation of Galerkin method yields to two additional stiffness matrices, K_2 and K_3, which provide superior performances to the method w.r.t. its straightforward version.

6 Superiority of Bastatsky/Khvoles Method

The first three frequencies of vibration in the $x-y$ and $x-z$ planes, computed by using the exact formulation are shown in Table 3.

Exact Solution [rad/sec]		
Mode	$x-y$ Plane	$x-z$ Plane
1	342.4121	67.5133
2	2166.4943	423.9471
3	6143.9243	1191.0450

Table 3: Exact solution

The rigorous Galerkin method, for 1, 2, 3, 25, 50, 75 and 100 terms, leads to the frequencies in Table 4 for the frequencies of vibration in the $x-y$ and $x-z$ plane, respectively. The relative error between the natural frequencies computed via the Galerkin method and the exact ones, computed as:

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	532.3005	525.2584	525.2059	385.5929	362.8573	353.8210	352.2366
2	-	3303.9364	3302.5800	2590.9190	2296.1458	2238.5734	2229.1627
3	-	-	9288.0450	7360.6548	6511.9947	6377.2805	6322.6071

Table 4: Frequencies of vibration for the x-y plane obtained with rigorous Galerkin method

$$\varepsilon = \frac{\omega_{Galerkin} - \omega_{Exact}}{\omega_{Exact}} \cdot 100\% \tag{32}$$

is reported in Table 5.

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	53.41%	53.40%	53.38%	12.61%	5.97%	3.33%	2.87%
2	-	52.50%	52.44%	19.59%	5.98%	3.33%	2.89%
3	-	-	51.17%	19.80%	5.99%	3.80%	2.91%

Table 5: Relative error between the rigorous Galerkin method and the exact solution for the frequencies of vibration in the x-y plane.

Advantages of application of Bastatsky/Khvoles formulations are transparent. The rigorous method was applied to static problems (Elishakoff et al, 2019), buckling of stepped columns (Elishakoff et al, 2020), and vibrations problems (Elishakoff et al, 2021a, 2021b).

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