QUASICONFORMAL WHITNEY PARTITION

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Abstract

Dedicated to the memory of Professor Alexanderv Khvoles
Whitney partition is a very important concept in modern geometric analysis. We discuss here a quasiconformal version of Whitney partition that can be useful for Sobolev spaces.

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1 Introduction

We start with a definition of a Whitney partition of domains in $\mathbf{R}^n$. Classical Whitney partition is a partition of a bounded domain Ω into diadic cubes with disjoint interiors and edges comparable to the distance to $\partial\Omega$. Its modern generalization that is called a Whitney partition is a partition into convex polyhedra with similar properties. Let us give a more accurate definition (see, for example [1]).

Let Ω be a bounded domain of $\mathbf{R}^n$ and $\Lambda = \mathbf{R}^n \setminus \Omega$. Let E_i be a family of convex closed n-dimensional polyhedra in $\mathbf{R}^n$, disjoint from Λ, covering Ω and with pairwise disjoint interiors. We will also need these polyhedra to have uniformly bounded ratio K_A of their exterior to interior radii. By the interior radius of a set A with nonempty interior we mean the greatest radius r_A of a ball contained in A; similarly the exterior radius of a set A is the smallest radius R_A of a ball containing A. The ratio K_A will be called the dilatation of A.

Such families of polyhedra with uniformly bounded dilatation will be called uniformly regular. We will also demand that the edges of these polyhedra are long, i.e. they have lengths uniformly comparable to the diameter of the polyhedron. As an example we can take a family of dyadic cubes disjoint from Λ. The diameter of E_i will be denoted by $\delta(E_i)$. The set Λ will be called the residual set of the family $\{E_i\}$ and $K_{int} := \sup_i K_{E_i}$ will be called the interior dilatation of the family $\{E_i\}$.

There are two conditions which are usually imposed on such families:

1. $\delta(E_i) \geq C \operatorname{dist}(E_i, \Lambda)$,
2. $\delta(E_i) \leq C^{-1} \operatorname{dist}(E_i, \Lambda)$.

If a family E_i satisfies both of these conditions, it is called a Whitney family.

We generalize this definition for a more general setting under additional assumption that the residual set $\mathbf{R}^{\mathrm{n}} \setminus \Omega'$ is unbounded and connected. We will call such domains as simple domains.

Definition.(Rough Whitney family). *Let Ω be a bounded subset of $\mathbf{R}^{\mathrm{n}}$, $\Lambda = \mathbf{R}^{\mathrm{n}} \setminus \Omega$ and $\{E_i\}$ be a family of closed sets in $\mathbf{R}^{\mathrm{n}}$ with nonempty interiors $\mathrm{V_i} = \mathrm{Int}(E_i)$ and the closures of V_i coinciding with E_i. The sets E_i are disjoint from Λ, covering Ω and their interiors V_i are pairwise disjoint. The family $\{E_i\}$ has bounded interior dilatation.*

The family E_i should also satisfy two geometric regularity conditions for some constant $C > 0$:

1.$\delta(E_i) \geq C \operatorname{dist}(E_i, \Lambda)$,

2.$\delta(E_i) \leq C^{-1} \operatorname{dist}(E_i, \Lambda)$.

If a family E_i satisfies all these conditions, we will call it as a rough Whitney family.

The smallest possible constant C will be called the exterior dilatation K_{ext} of the rough Whitney family.

Our main result connects concepts of Whitney families and rough Whitney families.

Theorem. *For any simple domain Ω quasiconformal image of any its Whitney family is a rough Whitney family.*

More detailed formulation with corresponding estimates will be discussed later.

We prove this result with the help of classical estimates of a corresponding capacity.

A two dimensional version of this result can be found in [3].

2 Quasiconformal Whitney family

A well-ordered triple $(F_0, F_1; \Omega)$ of nonempty sets, where Ω is an open set in $\mathbf{R}^{\mathbf{n}}$, and F_0, F_1 are closed subsets of $\overline{\Omega}$, is called a condenser on the Euclidean space $\mathbf{R}^{\mathbf{n}}$.

The value

$$\mathrm{Cap_p}(E) = \mathrm{Cap_p}(F_0, F_1; \Omega) = \inf \int\limits_{\Omega} |\nabla v|^p dx,$$

where the infimum is taken over all Lipschitz non-negative functions $v : \overline{\Omega} \to \mathbf{R}$, such that $v = 0$ on F_0, and $v = 1$ on F_1, is called p-capacity of the condenser $E = (F_0, F_1; \Omega)$. For $p = n$ it is the classical conformal capacity. We will use notation $\mathrm{Cap}(F_0, F_1; \Omega)$ for the conformal capacity. Of course the set of admissible functions can be empty. In this case $\mathrm{Cap_p}(F_0, F_1; \Omega) = \infty$.

For $1 < p$ for a finite value of p-capacity $0 \leq \mathrm{Cap}_p(F_0, F_1; \Omega) < +\infty$ there exists a unique continuous weakly differentiable function u_0 (an extremal function) such that:

$$\mathrm{Cap}_p(F_0, F_1; \Omega) = \int\limits_{\Omega} |\nabla u_0|^p dx.$$

Definition. *Let $\varphi : \Omega \to \Omega\prime$ be a homeomorphism between two domains in $\mathbf{R^n}$, $n \geq 2$. Then φ is said to be Q-quasiconformal, $Q \geq 1$, if*

$$Q^{-1}\mathrm{Cap}(F_0, F_1; \Omega) \leq \mathrm{Cap}(\varphi\,(F_0)\,, \varphi\,(F_1)\,; \Omega') \leq Q\,\mathrm{Cap}(F_0, F_1; \Omega)$$

for any condenser $E = (F_0, F_1; \Omega)$.

The minimal possible constant Q will be called the (quasiconformal) dilatation of φ.

This geometric definition is a global requirement that quickly yields many important properties of quasiconformal mappings. For example, the inverse of a quasiconformal mapping is automatically quasiconformal, quasiconformal mappings are weakly differentiable and its weak derivatives are integrable in degree n, etc...

For any Q-quasiconfromal homeomorphism $\varphi : \mathbf{R^n} \to \mathbf{R^n}$ images of closed balls have uniformly bounded interior dilatation K_{int} that depends only on n and Q [4]. Any Q-quasiconformal image of closed balls $\bar{B}(0, r)$ will be called the closed Q-quasiconformal ball or Q-quasiball. For example any convex polyhedra is a closed Q-quasiconfromal ball. The constant Q depends on n and its interior dilatation. This collection of known facts can be formalized as

Proposition 1. ([4]) *For any Q-quasiconfromal homeomorphism $\varphi : \mathbf{R^n} \to \mathbf{R^n}$ and any closed ball $B := B(x_0, r)$ the interior dilatation $K_{B,\Omega}$ is less or equal to a constant $C(Q, n)$ that depends only on the dilatation Q and n.*

This proposition will be generalized for balls in domains under an additional conditions for balls.

We will use notations: $B(x_0, r)$ is an open ball of radius r and center x_0 and $\bar{B}(x_0, r)$ is its closure.

Definition. *Let Ω be a domain in $\mathbf{R^n}$ and a closed ball $\overline{B}(x_0, r) \subset \Omega$. Choose such concentric open ball $B(x_0, K_{B(x_0,r),\Omega} r) \subset \Omega$ that its closure $\bar{B}(x_0, K_{B(x_0,r),\Omega} r)$ intersects $\partial\Omega$. We will call the constant $K_{B(x_0,r),\Omega} \in (1, \infty]$ the embedding coefficient of the ball $\bar{B}(x_0, r)$ in Ω.*

The embedding coefficient of any ball in $\mathbf{R^n}$ is ∞. In section 4 we will prove the following property of quasiconformal homeomorphisms.

Proposition 2. *Let $\varphi : \Omega \to \mathbf{R}^n$ be a Q-quasiconformal homeomorphism of a domain $\Omega \subset \mathbf{R}^n$ and $\overline{B}(x_0, r)$ is a closed ball with the embedding coefficient $K_{B(x_0,r),\Omega}$. Then the interior dilatation $K_{\varphi(B(x_0,r))}$ of its image $\varphi(B(x_0, r)$ depends only on $Q, K_{B(x_0,r),\Omega}$ and n.*

We will call a closed set E the closed relative Q-quasiball if there exist a domain $\Omega \subset \mathbf{R}^n$, a closed ball $\overline{B}(x_0, r)$ and a Q-quasiconformal homeomorphism $\varphi : \Omega \to \mathbf{R}^n$ such that $E = \varphi\left(\overline{B}(x_0, r)\right)$. We let K_E denote the embedding coefficient $K_{B(x_0,r),\Omega}$.

Definition (Q-quasiconfromal Whitney family). *Let Ω be a bounded domain in $\mathbf{R}^n$ and E_i be a family of closed relative Q-quasiballs in $\mathbf{R}^n$. The sets E_i are disjoint from Λ, covering Ω, with pairwise disjoint interiors V_i and their embedding coefficients $K_i := K_{E_u}$ are uniformly bounded.*

If a family E_i satisfies this conditions, it is called a Q-quasiconformal Whitney family.

Because any convex polyhedra is a quasiconformal image of the unit ball, any Whitney family is a Q-quasiconformal Whitney family. Of course a rough Whitney family is not necessary a Q-quasiconformal Whitney family.

3 Classical estimates of conformal capacity

We will need two estimates of the conformal capacity (see, for example [4, 2]). Because proofs are very simple we reproduce here slight modifications of both estimates, using classical embedding theorems.

Choose an open ball $B(x_0, r)$ an open ball of radius r and center x_0 and a positive constant $C_0 > 1$. Denote $R := C_0\, r$.

Lemma 1. $\operatorname{Cap}\left(\bar{B}(x_0, r), \mathbf{R}^n \setminus B(x_0, R); \mathbf{R}^n\right) = \omega_{n-1} \left[\ln (C_0)\right]^{1-n}.$
Here $\omega_{n-1}$is the volume of the unit $(n-1)$-dimensional sphere S^{n-1}.

Proof. Let $u : \mathbf{R}^n \to \mathbf{R}$ be an admissible smooth function for condensor $E = \left(\bar{B}(x_0, r), \mathbf{R}^n \setminus B(x_0, R); \mathbf{R}^n\right)$. By definition of an admissible function $u(x) \equiv 0$ on $\bar{B}(x_0, r)$ and $u(x) \equiv 1$ on $\mathbf{R}^n \setminus B(x_0, R)$. Using Hölder inequality for the spherical coordinate system (ρ,σ) we have

$$1 \le \int_r^R \nabla u(\rho,\sigma)d\rho \le \left(\int_r^R |\nabla u(\rho,\sigma)|^n \rho^{n-1}d\rho\right)^{1/n} \left(\int_r^R \rho^{-1}d\rho\right)^{\frac{n-1}{n}}.$$

Here $\rho := |x|$ and σ are spherical coordinates on the unit sphere $S^{n-1} := S^{n-1}(0,1)$. By previos calculations

$$\frac{1}{[\ln(C_0)]^{n-1}} = \frac{1}{\left[\ln\left(\frac{R}{r}\right)\right]^{n-1}} \le \int_r^R |\nabla u(\rho,\sigma)|^n \rho^{n-1}d\rho$$

for any $\sigma \in S^{n-1}$.

Integrating this inequality over S^{n-1} we obtain finally

$$\frac{\omega_{n-1}}{[\ln(C_0)]^{n-1}} \le \int_{S^{n-1}}\int_r^R |\nabla u(\rho,\sigma)|^n \rho^{n-1}d\rho = \int_{\mathbf{R}^n} |\nabla u(x)|^n\, dx$$

where ω_{n-1} is area of S^{n-1}.

The function $u_0(x) := \ln\left(\frac{R}{|x|}\right)[\ln(C_0)]^{-1}$ for $r \le |x| \le R$, 0 for any $|x| \ge R$ and 1 for any $|x| \le r$ is the extremal function for condensor E. It follows from direct calculations:

$$\int_{\mathbf{R}^n} |\nabla u_0(x)|^n\, dx = \omega_{n-1}[\ln(C_0)]^{-n}\int_r^R \rho^{-1}d\rho = \omega_{n-1}[\ln(C_0)]^{1-n}.$$

Therefore

$$\mathrm{Cap}\left(\bar{B}(x_0,r),\mathbf{R}^{\mathrm{n}} \setminus B(x_0,R);\mathbf{R}^{\mathrm{n}}\right) = \omega_{n-1}[\ln(C_0)]^{1-n}.$$

We will call a connected closed set a continuum. Choose two concentric $(n-1)$-dimensional spheres $S^{n-1}(0,r)$ and $S^{n-1}(0,R)$, $R \ge r$ and two continua F_0 and F_1 that join spheres. We shall use notations $R = C_0\, r$, $C_0 > 1$ and $D_{r,R} := B(0,R)\,\bar{\setminus}\, B(0,r)$. In the next lemma we will prove a below estimate of conformal capacity of condensor $E = (F_0, F_1, \mathbf{R}^{\mathrm{n}})$ using embedding theorems for the unit sphere.

Lemma 2. $\mathrm{Cap}(F_0,F_1;\mathbf{R}^{\mathrm{n}}) \ge C(n)\,\ln(C_0)$, *where a constant* $C(n)$ *depends on* n *only.*

Proof. Because continua F_0, F_1 join spheres $S^{n-1}(0,r)$ and $S^{n-1}(0,R)$ intersections $F_{0,\rho} := S^{n-1}_\rho \cap F_0$ and $F_{1,\rho} := S^{n-1}_\rho \cap F_1$ are not empty for any sphere $S^{n-1}_\rho := S^{n-1}(0,\rho)$, $r \le \rho \le R$. Any function u admissible for the conformal capacity of the condensor E is also admissible for conformal capacity of any condensor $E_\rho = (F_{0,\rho}, F_{1,\rho}, \mathrm{S}^{\mathrm{n}-1}_\rho)$ in the sphere S^{n-1}_ρ. By elementary calculations in the spherical coordinates ρ,σ we have

$$\int_r^R \left(\int_{S^{n-1}_\rho} |\nabla u(\sigma,\rho)|^n\, d\sigma\right) d\rho \le \int_{D_{R,r}} |\nabla u(x)|^n\, dx \le \int_{\mathbf{R}^{\mathrm{n}}} |\nabla u(x)|^n\, dx$$

where $D_{R,r}$ is a closed ring between spheres $S^{n-1}(0,r)$ and $S^{n-1}(0,R)$.

Using similarities $\varphi_\rho(x) = \rho x$ we get

$$\int_{S_\rho^{n-1}} |\nabla u(\sigma,\rho)|^n \, d\sigma = \frac{1}{\rho} \int_{S^{n-1}(0,1)} |\nabla \tilde{u}(\sigma)|^n \, d\sigma,$$

where $\tilde{u}(x) = u(\varphi_\rho(x))$.

By the classical Sobolev inequality for the unit sphere S^{n-1} we have

$$1 = \left\|\tilde{u}|L_\infty(S^{n-1})\right\|^n \leq K(n) \int_{S^{n-1}(0,1)} |\nabla \widetilde{u}(\sigma)|^n \, d\sigma$$

where constant $K(n)$ depends on n only.

Combining these two estimates we obtain

$$1 = \left\|u|L_\infty(S_\rho^{n-1})\right\|^n = \left\|\tilde{u}|L_\infty(S^{n-1}(0,1))\right\|^n \leq \rho K(n) \int_{S_\rho^{n-1}} |\nabla u(\sigma,\rho)|^n \, d\sigma.$$

Dividing by ρ and integrating we finally get

$$\frac{\ln(C_0)}{K(n)} = \frac{1}{K(n)} \int_r^R \frac{d\rho}{\rho} \leq \int_r^R \left(\int_{S_\rho^{n-1}} |\nabla u(\sigma,\rho)|^n \, d\sigma \right) d\rho$$

$$\leq \int_{\mathbf{R}^\mathrm{n}} |\nabla u(x)|^n \, dx$$

for any admissible function u of condensor E. By definition of conformal capacity

$$\frac{\ln(C_0)}{K(n)} \leq \mathrm{Cap}\,(F_0, F_1; \mathbf{R}^\mathrm{n})\,.$$

4 Local estimates of dilatations

Let $\varphi : \Omega \to \Omega'$ be a Q-quasiconformal homeomorphism of a domain $\Omega \subset \mathbf{R}^\mathrm{n}$ onto a domain $\Omega' \subset \mathbf{R}^\mathrm{n}$. Choose a closed ball $\bar{B}(x_0,r) \subset \Omega$ with the embedding coefficient $C_r := K_{B\{x_0,r\},\Omega}$.

Our goal is to prove for $F_r := \varphi(\bar{B}(x_0,r))$ the following two inequalities:

1. There exists a constant $C_1(\mathrm{Q},\mathrm{C_r},\mathrm{n})$ such that

$$K_{int}(F_r) \leq C_1(\mathrm{Q},\mathrm{C_r},\mathrm{n})$$

2. There exists a constant $C_2(\mathrm{Q},\mathrm{C_r},\mathrm{n})$ such that

$$C^{-1}\,\mathrm{dist}\left(F_r,\Omega'\right) \leq \delta\left(F_r\right) \leq C\,\mathrm{dist}\left(F_r,\Omega'\right)$$

,

Constants C_1, C_2 depend only on Q, C_r, n and do not depend on choice of domains Ω, Ω' and the Q-quasiconformal homeomorphism φ.

Proposition 3. *Suppose that Ω,Ω' are domains in $\mathbf{R}^n$ and the residual set $\Lambda = \mathbf{R}^n \setminus \Omega$ of Ω is unbounded and connected. Let $\varphi : \Omega \to \Omega'$ be a Q-quasiconformal homeomorphism of a domain $\Omega \subset \mathbf{R}^{\mathrm{n}}$ onto a domain $\Omega' \subset \mathbf{R}^{\mathrm{n}}$ and $\bar{B}(x_0, r) \subset \Omega$ be a closed ball with the embedding coefficient $K_{B(x_0,r),\Omega}$. Then there exists such positive constant $C_1(Q, C_r, n)$ that $K_{int}(F_r) \leq C_1(Q, K_{B(x_0,r),\Omega}, n)$.*

The constant $C_1(Q, K_{B(x_0,r),\Omega}, n)$ depends on $Q, K_{B(x_0,r),\Omega}$ and n only.

Proof. Denote by $\bar{B}_r := \bar{B}(y_0, \bar{r}) \subset F_r$ a closed ball of a maximal radius that belongs to $F_r := \varphi(B(x_o, r))$ and has the center at a point $y_0; = \varphi(x_0)$, by $B_R := B(y_0, \bar{R})$ an open ball of a minimal radius whose closure contains F_r, by $CB_R := \mathbf{R}^{\mathrm{n}} \setminus B_R$, by K_r the interior dilatation $K_{int}(F_r)$ and by C_r the embedding coefficient $K_{B(x_0,r),\Omega}$. By definition of the interior dilatation $K_r \leq \frac{\bar{R}}{\bar{r}}$.

By definition of the conformal capacity and previous Lemma

$$\operatorname{Cap}(B_R, \bar{\Omega}' \setminus B, \Omega') \leq \operatorname{Cap}(B_r, CB_R, \mathbf{R}^{\mathrm{n}}) = \omega_{n-1} \left[\ln\left(\frac{\bar{R}}{\bar{r}}\right)\right]^{1-n} \tag{1}$$

$$\leq \omega_{n-1} \left[\ln\left(K_r\right)\right]^{1-n}. \tag{2}$$

Because a homeomorphism φ^{-1} is Q-quasiconformal we have for two compact sets $F_0 := \varphi^{-1}(\bar{B}_r) \subset \bar{B}(x_0, r)$ and $F_1 = \varphi^{-1}(\bar{\Omega}' \setminus B_R) \subset \Omega \setminus B(x_0, r)$ the following inequality

$$\operatorname{Cap}(F_0, F_1, \Omega) \leq Q \operatorname{Cap}(\bar{B}_r, \bar{\Omega}' \setminus B_R, \Omega') \leq \omega_{n-1} \left[\ln\left(K_r\right)\right]^{1-n}. \tag{3}$$

Let us estimate $\operatorname{Cap}(F_0, F_1, \Omega)$ with the help of Lemma 2. By construction both sets F_0, F_1 have nonempty intersections S_0, S_1 with the sphere $S(x_0, r)$. We distinguish two different cases:

1. $\operatorname{dist}(S_0, S_1) \leq \min\left((C_r - 1)\frac{r}{2}, \frac{r}{2}\right)$;
2. $\operatorname{dist}(S_0, S_1) > \min\left((C_r - 1)\frac{r}{2}, \frac{r}{2}\right)$.

Let us use a short notation $\bar{r} := \min\left((C_r - 1)\frac{r}{2}, \frac{r}{2}\right)$.

Choose points $y_o \in S_0$ and $y_1 \bar{\in} S_1$ such that $\operatorname{dist}(S_0, S_1) = |y_1 - y_0|$. Let $\tilde{y} := y_0 + \frac{y_1 - y_0}{2}$ and $B_1 := \bar{B}(\tilde{y}, \bar{r})$ be a closed ball with center at $\tilde{y}$.

In the first case $|y_1 - y_0| \leq \bar{r}$ and continua F_0, F_1 intersect any sphere $S(\tilde{y}, \rho)$ for any $\bar{r} \leq \rho \leq 2\bar{r}$. By Lemma 2

$$\operatorname{Cap}\left(\mathrm{F}_0, \mathrm{F}_1; \Omega\right) \geq \operatorname{Cap}\left(\mathrm{F}_0 \cap \mathrm{B}_1, \mathrm{F}_1 \cap \mathrm{B}_1; \mathrm{B}_1\right) \geq C(n)\ \ln 2. \tag{4}$$

Combining this inequality and inequality 3 we obtain finally

$$C(n)\ \ln 2 \leq \omega_{n-1} \left[\ln\left(K_r\right)\right]^{1-n}. \tag{5}$$

The proposition is proved for the first case when

$$\operatorname{dist}(S_0, S_1) \leq \min\left((C_r - 1)\frac{r}{2}, \frac{r}{2}\right).$$

In the second case $\operatorname{dist}(S_0, S_1) \leq \min\left((C_r - 1)\frac{r}{2}, \frac{r}{2}\right)$ we have $|y_1 - y_0| > \bar{r}$. By definition of capacity

$$\operatorname{Cap}(F_0, F_1; \Omega) \geq \operatorname{Cap}(F_0 \cap B(x_0, C_r r), F_1 \cap B(x_0, C_r r); B(x_0, C_r r)).$$

There exists a quasiconformal homeomorphism ψ that maps $B(x_0, C_r r)$ onto itself, maps any sphere $S(x_0, \rho)$, $0 < \rho < C_r r$ onto itself and satisfies conditions: $\psi(y_1) = y_1$, and image of y_0 is the point $y_0^{\perp} := \psi(y_0)$ opposite to y_0 on $S(x_0, r)$. The coefficient of quasiconformality Q_1 of ψ can be easily estimated $Q_1 \leq \frac{\pi r}{\bar{r}} = \frac{2}{\min(C_r - 1, 1)}$.

By definition of quasiconformal homeomorphism we have

$$\operatorname{Cap}(F_0 \cap B(x_0, C_r r), F_1 \cap B(x_0, C_r r); B(x_0, C_r r)) \geq$$
$$\frac{1}{Q_1} Cap\,(\psi(F_0) \cap B(x_0, C_r r), \psi(F_1) \cap B(x_0, C_r r); B(x_0, C_r r)).$$

Choose two closed balls

$$B_2 := \bar{B}\left(x_0 + \frac{y_0^{\perp} - x_0}{2}, \frac{r}{2}\right)$$

and

$$B_3 := \bar{B}\left(x_0 + \frac{y_0^{\perp} - x_0}{2}, \frac{r}{2} + (C_r - 1)r\right) \subset B(x_0, C_r r).$$

By construction continua $\psi(F_0)$ and $\psi(F_1)$ intersect any sphere $S\left(x_0 + \frac{y_0^{\perp} - x_0}{2}, \rho\right)$ for $\frac{r}{2} \leq \rho \leq \frac{r}{2} + (C_r - 1)r$. Therefore by Lemma 2

$$\operatorname{Cap}(\psi(F_0) \cap B(x_0, C_r r), \psi(F_1) \cap B(x_0, C_r r); B(x_0, C_r r))$$
$$\geq C(n) \ln(1 + 2(C_r - 1)).$$

Finally we have

$$\operatorname{Cap}(F_0, F_1; \Omega) \geq C(n) \min(C_r - 1, 1) \ln(1 + 2(C_r - 1)). \tag{6}$$

Combining this inequality and inequality 3 we obtain finally

$$C(n) \min(C_r - 1, 1) \ln(1 + 2(C_r - 1)) \leq \omega_{n-1} \left[\ln(K_r)\right]^{1-n}. \tag{7}$$

The proposition is proved for the second case:

$$\operatorname{dist}(S_0, S_1) > \min\left((C_r - 1)\frac{r}{2}, \frac{r}{2}\right)$$

Recall that K_r is the short notation for $K_{int}(F_r)$. Combining inequalities (3, 4, 6) we obtain finally the constant $C_1(Q, C_r, n)$

$$C(n)\min\left[\ln 2, \min(C_r-1,1)\ln\left(1+2(C_r-1)\right)\right] \le \omega_{n-1}\left[\ln\left(K_r\right)\right]^{1-n}$$

i.e.

$$K_r \le \exp\left[\left\{\frac{\omega_{n-1}}{C(n)\min\left[\ln 2, \min(C_r-1,1)\ln\left(1+2(C_r-1)\right)\right]}\right\}^{\frac{1}{n-1}}\right].$$

Remark. *We proved a stronger estimate:*

$$\frac{\bar{R}}{\bar{r}} \le \exp\left[\left\{\frac{\omega_{n-1}}{C(n)\min\left[\ln 2, \min(C_r-1,1)\ln\left(1+2(C_r-1)\right)\right]}\right\}^{\frac{1}{n-1}}\right].$$

Let us return to estimates of $\delta\left(F_r\right)$ with the help of $\operatorname{dist}\left(F_r, \partial\Omega'\right)$ i.e. to estimates of $K_{ext}(F_r)$. We start with the upper estimate:

Proposition 4. *Suppose that Ω,Ω' are domains in $\mathbf{R}^n$ and the residual set $\Lambda = \mathbf{R}^n \setminus \Omega'$ of Ω' is unbounded and connected. Let $\varphi : \Omega \to \Omega'$ be a Q-quasiconformal homeomorphism of a bounded domain $\Omega \subset \mathbf{R}^{\mathrm{n}}$ onto a bounded domain $\Omega' \subset \mathbf{R}^{\mathrm{n}}$, $\bar{B}(x_0, r) \subset \Omega$ be a closed ball with the embedding coefficient $K_{B(x_0,r),\Omega}$. Then*

$$\delta\left(F_r\right) \le C_1(\mathrm{Q}, \mathrm{C_r}, \mathrm{n})\operatorname{dist}\left(F_r, \Lambda'\right) = C_1\operatorname{dist}\left(F_r, \partial\Omega'\right)$$

where a constant $C_2 := C_2(Q, C_r, n)$ depends only on Q, C_r, n.

Proof. We will use notations: $F_r = \varphi(\bar{B}(x_0, r)), K_r$ for the interior dilatation $K_{int}(F_r)$, C_r for the embedding coefficient $K_{B(x_0,r),\Omega}$ and $F_R := \overline{\Omega}' \setminus \varphi(B(x_0, C_r r))$.

Because $\varphi : \Omega \to \Omega'$ is Q-quasiconformal

$$\operatorname{Cap}(F_r, F_R; \Omega') \le Q \operatorname{Cap}(\bar{B}(x_0, r), \overline{\Omega} \setminus B(x_0, C_r r)); \Omega).$$

By definition of the conformal capacity

$$\operatorname{Cap}(\bar{B}(x_0, r), \overline{\Omega} \setminus B(x_0, C_r r)); \Omega) = \operatorname{Cap}(\bar{B}(x_0, r), \mathbf{R}^{\mathrm{n}} \setminus B(x_0, Cr)); \mathbf{R}^{\mathrm{n}}).$$

By Lemma 1

$$\operatorname{Cap}(\bar{B}(x_0, r), \mathbf{R}^{\mathrm{n}} \setminus B(x_0, Cr)); \mathbf{R}^{\mathrm{n}}) = \omega_{n-1}\left[\ln\left(C_r\right)\right]^{1-n}.$$

Hence

$$\operatorname{Cap}(F_r, F_R; \Omega') \le \omega_{n-1}\left[\ln\left(C_r\right)\right]^{1-n}$$

Choose points $y_o \in F_r$and $y_1 \in \partial\Omega'$ such that $\operatorname{dist}(F_r, \partial\Omega') = |y_1 - y_0|$. Let $\tilde{y} := y_0 + \frac{y_1 - y_0}{2}$ and $\bar{B}(\tilde{y}, \rho_1)$ be a closed ball with center at $\tilde{y}$.

By Proposition 3 F_r contains a closed ball of a radius $\rho_2 := \frac{1}{2} K_r \delta(F_r)$. Therefore both continuums F_r and F_R intersect any sphere $S(\tilde{y}, \rho)$ for any $\rho \in [\rho_1, \rho_1 + \rho_2]$. By Lemma 2 we have

$$\operatorname{Cap}(F_r, F_R; \Omega') \geq \operatorname{Cap}(F_r, F_R; \bar{B}(\tilde{y}, \rho_2) \setminus B(\tilde{y}, \rho_1)) \geq$$
$$C(n) \ln \left(\tfrac{\operatorname{dist}(F_r, \partial\Omega') + K_r \delta(F_r)}{\operatorname{dist}(F_r, \partial\Omega')} \right).$$

Combining both estimates for $\operatorname{Cap}(F_r, F_R; \Omega')$ we obtain finally

$$\mathrm{C(n)} \ln \left(\frac{\mathrm{dist(F_r, \partial\Omega') + K_r \delta(F_r)}}{\mathrm{dist(F_r, \partial\Omega')}} \right) \leq \operatorname{Cap}(F_r, F_R; \Omega') \leq \omega_{n-1} \left[\ln(C_r) \right]^{1-n}.$$

It means that

$$\operatorname{dist}(F_r, \partial\Omega') + K_r \delta(F_r) \leq \operatorname{dist}(F_r, \partial\Omega') \exp \left[\frac{\omega_{n-1}}{C(n) \left[\ln(C_r) \right]^{n-1}} \right].$$

Because $C_r > 1$ the number under exponent is positive and the exponent value is bigger than 1. Therefore

$$\delta(F_r) \leq \left\{ K_r^{-1} \exp \left[\frac{\omega_{n-1}}{C(n) \left[\ln(C_r) \right]^{n-1}} \right] \right\} \operatorname{dist}(F_r, \partial\Omega').$$

The second estimate is an estimate from below with some additional restriction on the domain Ω'. We suppose additionally that any component of $\mathbf{R}^{\mathrm{n}} \setminus \Omega'$ is unbounded. We call such domains simple domains.

Proposition 5. *Suppose that Ω,Ω' are domains in $\mathbf{R}^n$ and the residual set $\Lambda = \mathbf{R}^n \setminus \Omega$ of Ω is unbounded and connected (i.e Ω is a simple domain). Let $\varphi : \Omega \to \Omega'$ be a Q-quasiconformal homeomorphism , $\bar{B}(x_0, r) \subset \Omega$ be a closed ball with embedding coefficient $K_{B(x_0,r),\Omega}$. Then*

$$\delta\left(F_r\right) \geq C_3(\mathrm{Q}, \mathrm{K_{B(x_0,r),\Omega}}, \mathrm{n}) \operatorname{dist}\left(F_r, \Lambda'\right)$$

where a constant $C_3 := C_3(Q, K_{B(x_0,r),\Omega}, n)$ depends only on $Q, K_{B(x_0,r),\Omega}, n$.

Proof. Denote by $\bar{B}_r := \bar{B}(y_0, \bar{r}) \subset F_r$ a closed ball of a maximal radius that belongs to $F_r := \varphi(B(x_o, r))$ and has the center at a point $y_0; = \varphi(x_0)$, by $B_R := B(y_0, \bar{R})$ an open ball of a minimal radius whose closure contains F_r, by $CB^R := \mathbf{R}^{\mathrm{n}} \setminus B^R$, by K_r the ratio $\overline{K}_r \leq \frac{\bar{R}}{\bar{r}}$ and by C_r the embedding coefficient $K_{B(x_0,r),\Omega}$. By definition of the interior dilatation $K_r \leq \overline{K}_r = \frac{\bar{R}}{\bar{r}}$.

By Proposition 3 $\overline{K}_r \leq \tilde{C}_r(Q, C_r, n) := \tilde{C}_r$. It means that $\delta\left(F_r\right) \leq 2\tilde{C}_r \bar{r}$. Let $B_{r_d} := B(y_0, r_d)$ be a greatest open ball such that $S(y_0, r_d) \cap$

$\partial\Omega' \neq \emptyset$. By construction $r_d \geq \overline{r} + dist\,(F_r, \partial\Omega')$. By Lemma 1 and monotonicity of the conformal capacity

$$\mathrm{Cap}(\bar{B}_r, \Omega' \setminus B_{r_d}; \Omega') = \mathrm{Cap}(\bar{B}_r, \mathbf{R}^{\mathrm{n}} \setminus B_{r_d}; \mathbf{R}^{\mathrm{n}}) = \omega_{n-1} \left[\ln \left(\frac{r_d}{\overline{r}} \right) \right]^{1-n}$$

$$\leq \omega_{n-1} \left[\ln \left(\frac{\overline{r} + dist\,(F_r, \partial\Omega')}{\overline{r}} \right) \right]^{1-n}.$$

Using monotonicity of conformal capacity and quasi-invariance of conformal capacity under Q-quasiconformal homeomorphisms we obtain

$$\mathrm{Q}^{-1}\mathrm{Cap}(\overline{\mathrm{B}}(\mathrm{x}_0, \mathrm{r}), \Lambda; \Omega) \leq \mathrm{Cap}(\bar{\mathrm{B}}_{\mathrm{r}}, \Lambda'; \Omega') \leq \mathrm{Cap}(\bar{\mathrm{B}}_{\mathrm{r}}, \mathbf{R}^{\mathrm{n}} \setminus \mathrm{B}_{\mathrm{r_d}}; \Omega')$$

$$\leq \mathrm{Cap}(\bar{B}_r, \overline{\Omega}' \setminus B_{r_d}; \Omega').$$

Choose a point $x_1 \in \partial\Omega$ closest to $\overline{B}(x_0, r)$ and two balls

$$B_1 := B\left(x_0 + \frac{x_1 - x_0}{2}, \frac{C_r - 1}{2}\right)$$

and

$$\overline{B}_2 := \overline{B}\left(x_0 + \frac{x_1 - x_0}{2}, \frac{C_r - 1}{2} + 2r\right)$$

. Continuums $F_0 := \overline{B}(x_0, r)$ and $F_1 := \Lambda \cap B\left(x_0 + \frac{x_1 - x_0}{2}, \frac{C_r - 1}{2} r + 2r\right)$ join spheres $S^{n-1}\left(x_0 + \frac{x_1 - x_0}{2}, \frac{C_r - 1}{2} r + 2r\right)$ and $B\left(x_0 + \frac{x_1 - x_0}{2}, \frac{C_r - 1}{2} r\right)$. By monotonicity of capacity and Lemma 2 we get

$$\mathrm{C(n)} \ln \left\{ 1 + \frac{1}{\mathrm{C_r} - 1} \right\} \leq \mathrm{Cap}(F_0, F_1; \overline{B}_2 \setminus B_1) \leq \mathrm{Cap}(\overline{B}(x_0, r), \Lambda; \Omega)$$

Combining all previous estimates we get

$$\ln \left(1 + \frac{1}{C_r - 1} \right) \leq \frac{Q\omega_{n-1}}{C(n)} \omega_{n-1} \left[\ln \left(\frac{\overline{r} + dist\,(F_r, \partial\Omega')}{\overline{r}} \right) \right]^{1-n}.$$

It means that

$$\left[\ln \left(1 + \frac{dist\,(F_r, \partial\Omega')}{\overline{r}} \right) \right]^{n-1} \leq \frac{Q\omega_{n-1}}{C(n)} \omega_{n-1} \left[\ln \left(1 + \frac{1}{C_r - 1} \right) \right]^{-1}$$

The constant

$$C_0(Q, C_r, n) := \frac{Q\omega_{n-1}}{C(n)} \omega_{n-1} \left[\ln \left(1 + \frac{1}{C_r - 1} \right) \right]^{-1}$$

is positive.

After elementary calculations we have

$$\frac{dist\,(F_r, \partial\Omega')}{\overline{r}} \leq \exp\left(C_0(Q, C_r, n)^{\frac{1}{n-1}}\right) - 1.$$

Recall that C_r is a short notation for $K_{B(x_0,r),\Omega}$ and denote

$$C_3(Q, K_{B(x_0,r),\Omega}, n) := \left[\exp\left(C_0(Q, C_r, n)^{\frac{1}{n-1}}\right) - 1\right]^{-}.$$

Because $\overline{r} \leq \delta(F_r)$ we rewrite the last inequality as

$$C_3(Q, K_{B(x_0,r),\Omega}, n)\, dist\left(F_r, \partial\Omega'\right) \leq \overline{r} \leq \delta(F_r)$$

The main result of this paper can be formulated in more general way.

Theorem. *For any simple domain Ω quasiconformal image of any its Whitney family is a rough Whitney family.*

Follows directly from Propositions 3,4,5.

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