THE BOUNDARY VALUE PROBLEMS OF THE THEORY OF ELASTICITY FOR A SPHERE AND FOR THE SPACE WITH SPHERICAL CAVITY WITH DOUBLE VOIDS STRUCTURE

Lamara Bitsadze

I. Vekua Institute of Applied Mathematics
of I. Javakhishvili Tbilisi State University
University str. 2, 0186 Tbilisi, Georgia
lamara.bitsadze@gmail.com

Abstract

The purpose of this paper is to construct explicit solutions of the Dirichlet type and the Neumann type boundary value problems of the theory of elasticity for a sphere and for a space with spherical cavity with a double voids structure. The solutions of considered boundary value problems are presented as absolutely and uniformly convergent series.

Keywords and phrases: Sphere with double voids structure, explicit solution, space with spherical cavity, absolutely and uniformly convergent series.

AMS subject classification (2010): 74G05, 74F10, 74F99.

1 Introduction

Porous materials have applications in many fields of engineering, such as the petroleum industry, material science and biology. This theory studies the behavior of elastic porous materials like the rock, the bone and the manufactured porous materials. Below, we will consider a few works, which give the main results and bibliographical data.

Theory of poroelasticity is presented in the works by Biot [1,2]. Later the non-linear version of elastic materials with voids was proposed by Nunziato and Cowin [3] and the linear version was developed by Cowin and Nunziato [4] to study mathematically the mechanical behavior of porous solids. By using the mechanics of materials with voids the theories of elasticity and thermoelasticity for materials with double porosity structure are presented by Ieşan and Quintanilla [5]. The basic equations of this theory involve the displacement vector field and the volume fraction fields associated with the pores and the fissures. Ieşan in [6] established a variational theory for thermoelastic materials with voids.

The basic results on the theory of porous materials can be found in the books ([7],[8],[9],[10]).

Many problems are investigated by several researchers in the elastic materials with voids, using the theory developed by Cowin and his co-workers, by applying different methods such as an analytical, numerical and the complex variable technique to investigate the two-dimensional and three-dimensional boundary value problems of the theory of elasticity and thermoelasticity. Some of these results are presented in [11-31] and in references therein.

The purpose of this paper is to construct explicit solutions of the Dirichlet type and the Neumann type boundary value problems of the theory of elasticity for a sphere and for a space with spherical cavity with a double voids structure. The solutions are presented as absolutely and uniformly convergent series.

2 Basic Equations and Formulation of the Problem

Let us assume that D is a ball of radius R centered at point $O(0,0,0)$ in the Euclidean 3D space E^3 and S is a spherical surface of radius R. Let D^- be the whole space with spherical cavity and with boundary S. Let $x = (x_1, x_2, x_3) \in E^3$. Let us assume that the domain $D(D^-)$ is composed of an isotropic homogeneous elastic material with double voids structure.

The basic system of linearized equations of motion in the theory of elasticity for homogeneous and isotropic materials with double voids structure can be written as [5]

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{graddiv}\mathbf{u} + b\mathrm{grad}\varphi + d\mathrm{grad}\psi = 0,\\
&(\alpha\Delta - \alpha_1)\varphi + (\beta\Delta - \alpha_3)\psi - b\,\mathrm{div}\mathbf{u} = 0, \qquad (1)\\
&(\beta\Delta - \alpha_3)\varphi + (\gamma\Delta - \alpha_2)\psi - d\,\mathrm{div}\mathbf{u} = 0,
\end{aligned}$$

where $\mathbf{u} := (u_1, u_2, u_3)^\top$ is the displacement vector in a solid, $\varphi(\mathbf{x})$ and $\psi(\mathbf{x})$ are the changes of volume fractions from the reference configuration corresponding to macro- and micropores, respectively, λ, μ, β, α,b, d, γ, α_1, α_2, α_3 are constitutive coefficients, Δ is the Laplacian operator. Throughout this paper the superscript "$\top$" stands for the transpose operation.

Definition: A vector-function $\mathbf{U} = (\mathbf{u}, \varphi, \psi)$ is called regular in the domain $D(D^-)$, if

$$\mathbf{U} \in C^2(D) \cap C^1(\overline{D}), \quad (\mathbf{U} \in C^2(D^-) \cap C^1(\overline{D^-})),$$

in the case of the domain D^-, $\mathbf{U}$ additionally should satisfy the following conditions at infinity:

$$\mathbf{U}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \qquad \frac{\partial \mathbf{U}}{\partial x_j} = O(|\mathbf{x}|^{-2}) \qquad |\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2 >> 1,$$

$$j = 1, 2, 3.$$

In the sequel, we consider the following boundary value problems (BVPs):

Problem 1: (The Dirichlet type BVP.) Find a regular solution $\mathbf{U} = (\mathbf{u}, \varphi, \psi)$ of the system (1) in the domain D, satisfying the following boundary conditions on S :

$$\mathbf{u}^+(\mathbf{z}) = \boldsymbol{F}^+(\mathbf{z}), \quad \varphi^+(\mathbf{z}) = f_4^+(\mathbf{z}), \quad \psi^+ = f_5^+(\mathbf{z}), \quad \mathbf{z} \in S.$$

Problem 2: (The Neumann type BVP.) Find a regular solution $\mathbf{U} = (\mathbf{u}, \varphi, \psi)$ of the system (1) in the domain D^- satisfying the following boundary conditions on S :

$$[\mathbf{PU}]^- = \boldsymbol{F}^-(\mathbf{z}), \quad \left[\frac{\partial \varphi}{\partial n}\right]^- = f_4^-(\mathbf{z}), \quad \left[\frac{\partial \psi}{\partial n}\right]^- = f_5^-(\mathbf{z}), \quad \mathbf{z} \in S.$$

The vector $\mathbf{PU}$ is defined in the following form

$$\mathbf{P}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{U} = \mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{u} + \mathbf{n}(b\varphi + d\psi), \tag{2}$$

where $\mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{u}$ is the stress vector in the classical theory of elasticity

$$\mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{u} = 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda \mathbf{n} \mathrm{div} \mathbf{u} + \mu[\mathbf{n} \cdot \mathrm{rot} \mathbf{u}],$$

the vector-function $\mathbf{F}(\mathbf{z}) = (f_1, f_2, f_3)$, and the functions $f_4(\mathbf{z})$, $f_5(\mathbf{z})$ are prescribed functions on S, at $\mathbf{z}$. The symbol $\mathbf{U}^+(\mathbf{U}^-)$ denotes the limits of $\mathbf{U}(\mathbf{x}) = (\mathbf{u}, \varphi, \psi)$ on $\mathbf{z} \in S$ from $D(D^-)$

$$\mathbf{U}^+(\mathbf{z}) = \lim_{D \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}), \quad \mathbf{U}^-(\mathbf{z}) = \lim_{D^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}).$$

Throughout this paper assume that $b^2 + d^2 > 0$ and the following inequalities are true [5]

$$\begin{gathered} \mu > 0, \;\; \alpha > 0, \;\; \alpha_0 = \alpha\gamma - \beta^2 > 0, \;\; \alpha_1 > 0, \;\; \beta_0 = \alpha_1\alpha_2 - \alpha_3^2 > 0, \\ (3\lambda + 2\mu)\beta_0 > 3\gamma_1, \;\; \mu_0 = \lambda + 2\mu > 0, \;\; \gamma_1 = \alpha_2 b^2 + \alpha_1 d^2 - 2bd\alpha_3. \end{gathered} \tag{3}$$

Theorem 1. *If the conditions (3) are satisfied, then the Problem 1 (the Problem 2) admits at most one regular solution in $D(D^-)$.*

Theorem 1 can be proved similarly, as the corresponding theorems are proved in [32],[33].

3 Preliminaries

In this section, we cite some basic theorems without proof, (its are proved in [30],) which are useful in our subsequent.

Theorem 2. *If $\boldsymbol{U} := (\boldsymbol{u}, \varphi, \psi)$ is a regular solution of the homogeneous system (1) then $\boldsymbol{u}$, $\operatorname{div}\boldsymbol{u}$, φ and ψ satisfy the conditions*[30]

$$\Delta\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\mathbf{u} = 0,$$

$$\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\mathbf{\Phi} = 0,$$

where $\mathbf{\Phi} = (\operatorname{div}\mathbf{u}, \varphi, \psi)$, λ_j^2, $j = 1, 2$ are roots of equation

$$\alpha_0\mu_0\xi^2 + A_1\xi + A_2 = 0,$$

$$A_1 = \mu_0\beta_2 - \beta_1, \quad \mu_0 = \lambda + 2\mu, \quad \beta_2 = \gamma\alpha_1 + \alpha\alpha_2 - 2\beta\alpha_3$$

$$\beta_1 = \gamma b^2 + \alpha d^2 - 2bd\beta, \quad A_2 = \mu_0\beta_0 - \gamma_1, \quad \beta_0 = \alpha_1\alpha_2 - \alpha_3^2$$

$$\alpha_0 = \gamma\alpha - \beta^2, \quad \gamma_1 = \alpha_2 b^2 + \alpha_1 d^2 - 2bd\alpha_3,$$

Theorem 3. *The regular solution $\boldsymbol{U} = (\boldsymbol{u}, \varphi, \psi)$ of system (1) admits in the domain of regularity a representation* [30]

$$\begin{aligned} \mathbf{u} &= \mathbf{\Psi} - \operatorname{grad}\left[(m_0 - 1)h_0 + \sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right], \\ \varphi &= B_0 h + \sum_{j=1}^{2} B_j h_j, \quad \psi = C_0 h + \sum_{j=1}^{2} C_j h_j, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\operatorname{div}\mathbf{u} = h + \sum_{j=1}^{2} h_j, \quad \Delta\mathbf{\Psi} = 0, \quad \operatorname{div}\mathbf{\Psi} = m_0 h, \quad m_0 = \frac{A_2}{\mu\beta_0}, \\ &\Delta h_0 = h, \quad \Delta h = 0, \quad (\Delta + \lambda_j^2)h_j = 0, \end{aligned}$$

$$
\begin{aligned}
&B_0 = \frac{d\alpha_3 - b\alpha_2}{\beta_0}, \quad C_0 = \frac{-d\alpha_1 + b\alpha_3}{\beta_0},\\
&B_j = \frac{B_0\beta_0 + d_1\lambda_j^2}{\delta_j}, \quad C_j = \frac{C_0\beta_0 + d_2\lambda_j^2}{\delta_j},\\
&d_1 = d\beta - b\gamma, \quad d_2 = b\beta - d\alpha, \quad bB_0 + dC_0 = -\frac{\gamma_1}{\beta_0},\\
&bB_J + dC_j = -\mu_0, \quad \delta_j = \frac{\gamma_1 + \beta_1\lambda_j^2}{\mu_0}.
\end{aligned} \tag{5}
$$

Let us introduce the spherical coordinates equalities

$$
\begin{aligned}
&x_1 = \rho\sin\vartheta\cos\eta, \quad x_2 = \rho\sin\vartheta\sin\eta, \quad x_3 = \rho\cos\vartheta, \quad x \in D,\\
&y_1 = R\sin\vartheta_0\cos\eta_0, \quad y_2 = R\sin\vartheta_0\sin\eta_0, \quad y_3 = R\cos\vartheta_0, \quad y \in S,\\
&|x|^2 = \rho^2 = x_1^2 + x_2^2 + x_3^2, \quad 0 \le \vartheta \le \pi, \quad 0 \le \eta \le 2\pi \quad 0 \le \rho \le R.
\end{aligned}
$$

$(\mathbf{x}\cdot\mathbf{g}) = \sum\limits_{k=1}^{3} x_k g_k$ and $[\mathbf{x}\cdot\mathbf{g}]$ denotes the usual scalar product and the vector product of the two vectors $\mathbf{x}$ and $\mathbf{g}$ respectively.
Operator $\dfrac{\partial}{\partial S_k(x)}$ is defined as follows:

$$
[\mathbf{x}\cdot\nabla]_k = \frac{\partial}{\partial S_k(x)}, \qquad k = 1, 2, 3, \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).
$$

Below we introduce the following identities:

$$
\begin{aligned}
&\frac{\partial}{\partial n}\mathrm{grad}g = \frac{1}{\rho}\mathrm{grad}\left(\rho\frac{\partial}{\partial\rho} - 1\right)g, \quad \mathrm{div}\mathbf{n}g = \left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right)g,\\
&\mathrm{div}[\mathbf{n}\cdot\mathrm{rot}\mathbf{u}] = -\frac{\partial}{\partial\rho}\mathrm{div}\mathbf{\Psi}, \;\; if \;\; \mathbf{\Delta\Psi} = 0, \quad (\mathbf{x}\cdot\mathrm{grad}) = \rho\frac{\partial}{\partial\rho},\\
&\mathrm{div}\frac{\partial\mathbf{u}}{\partial n} = \left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right)\mathrm{div}\mathbf{u} - \frac{1}{\rho}\left(\mathbf{n}\cdot\frac{\partial\mathbf{u}}{\partial n}\right),\\
&\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})} = \sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k(\mathbf{z})}, \quad \sum_{k=1}^{3}\frac{\partial(\mathbf{Pu})_k}{\partial S_k} = \mu\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k},
\end{aligned} \tag{6}
$$

$$\left(\mathbf{n}\cdot\frac{\partial\mathbf{u}}{\partial n}\right)=\frac{1}{\rho}\left(\mathbf{x}\cdot\frac{\partial\mathbf{\Psi}}{\partial n}\right)-\frac{\partial^2}{\partial\rho^2}\left[(m_0-1)h_0+\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right],$$

$$\left(\mathbf{x}\cdot\frac{\partial\mathbf{\Psi}}{\partial n}\right)=\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)(\mathbf{x}\cdot\mathbf{\Psi}).$$

We introduce the following functions

$$\begin{aligned}
&(\mathbf{x}\cdot\mathbf{F})^{\pm}=q_1^{\pm}(y),\quad \varphi^{+}=q_2^{+}(\mathbf{y}),\\
&\psi^{+}=q_3^{+}(\mathbf{y}),\quad (\mathrm{div}\mathbf{F})^{\pm}=q_4^{\pm},\quad \mathbf{y}\in S.\\
&\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})}=\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k(\mathbf{z})}=q_5^{+},\\
&\sum_{k=1}^{3}\frac{\partial(\mathbf{Pu})_k}{\partial S_k}=\mu\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k}=q_5^{-}.
\end{aligned}\tag{7}$$

In what follows we assume that the functions q_k, $k=1,..,5$, are representable in the form of the series

$$q_k^{\pm}(\mathbf{y})=\sum_{n=0}^{\infty}q_{kn}^{\pm}(\xi_0,\eta_0),\tag{8}$$

where $q_{kn}^{\pm}$ $k=1,..,5$ are the spherical harmonics of order n

$$q_{kn}^{\pm}=\frac{2n+1}{4\pi R^2}\int\limits_S P_n(\cos\gamma)q_k^{\pm}(\mathbf{y})dS_y,$$

P_n is Legender polynomial of the n-th order, γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma=\frac{1}{|\mathbf{x}||\mathbf{y}|}\sum_{k=1}^{3}x_k y_k.$$

Substituting (4) in (2) and using the identities (6), the vector $\mathbf{PU}$ takes the form

$$\mathbf{P}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{u}=2\mu\frac{\partial\mathbf{\Psi}}{\partial n}-2\mu\frac{1}{\rho}\mathrm{grad}\left(\rho\frac{\partial}{\partial\rho}-1\right)\left[(m_0-1)h_0+\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right]\tag{9}$$

$$+\mathbf{n}[\lambda\mathrm{div}\mathbf{u}+b\varphi+d\psi]+\mu[\mathbf{n}\cdot\mathrm{rot}\mathbf{\Psi}],$$

where

$$\lambda\mathrm{div}\mathbf{u}+b\varphi+d\psi=\mu(m_0-2)h-2\mu\sum_{n=0}^{2}h_j.$$

4 The Solution of Problem 1

We are looking for a solution of system (1) in the form (4). Taking into account the identity $(\mathbf{x}\cdot \mathrm{grad}) = \rho\dfrac{\partial}{\partial\rho}$, from (4) we obtain

$$(\mathbf{x}\cdot\mathbf{u}) = (\mathbf{x}\cdot\mathbf{\Psi}) - \rho\frac{\partial}{\partial\rho}\left[(m_0-1)h_0 + \sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right]. \tag{10}$$

On the basis of the identity

$$\Delta(gq) = g\Delta q + q\Delta g + 2\mathrm{grad}g\mathrm{grad}q,$$

for the function $(\mathbf{x}\cdot\mathbf{\Psi})$ we shall have

$$\Delta(\mathbf{x}\cdot\mathbf{\Psi}) = 2\mathrm{div}\mathbf{\Psi} = 2m_0h,$$

the general solution of which is defined as a sum of homogeneous and particular solutions of nonhomogeneous equation ,respectively

$$(\mathbf{x}\cdot\mathbf{\Psi}) = \Omega + 2m_0h_0, \tag{11}$$

where Ω is an arbitrary harmonic function $\Delta\Omega = 0$ and $\Delta h_0 = h$.

From (4), (10) and (11), we get

$$\begin{aligned}
&(\mathbf{x}\cdot\mathbf{u}) = \Omega + 2m_0h_0 - \rho\frac{\partial}{\partial\rho}\left[(m_0-1)h_0 + \sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right],\\
&\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(\mathbf{z})} = \sum_{k=1}^{3}\frac{\partial \Psi_k}{\partial S_k(\mathbf{z})},
\end{aligned}\tag{12}$$

$$\varphi = B_0h + \sum_{j=1}^{2}B_jh_j, \quad \psi = C_0h + \sum_{j=1}^{2}C_jh_j, \quad \mathrm{div}\mathbf{u} = h + \sum_{j=1}^{2}h_j.$$

Let us assume that functions h, Ω, $\sum_{k=1}^{3}\dfrac{\partial\Psi_k}{\partial S_k(\mathbf{z})}$ and h_j $(j = 1,2)$ are sought in the form [34],

$$\begin{aligned}
&h(\mathbf{x}) = \sum_{n=0}^{\infty}\frac{\rho^n}{R^n}Z_n(\vartheta,\eta), \quad \Omega = \sum_{n=0}^{\infty}\frac{\rho^n}{R^n}Z_{1n}(\vartheta,\eta),\\
&h_j(\mathbf{x}) = \sum_{n=0}^{\infty}\phi_n(\lambda_j\rho)Y_{jn}(\vartheta,\eta), \quad j=1,2,\\
&\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k(\mathbf{z})} = \sum_{n=0}^{\infty}\frac{\rho^n}{R^n}Z_{2n}(\vartheta,\eta), \quad \rho < R,
\end{aligned}\tag{13}$$

where Z_n Z_{jn} *and* Y_{jn} $j = 1, 2$ are the unknown spherical harmonic of order n,

$$\phi_n(\lambda_j \rho) = \frac{\sqrt{R} J_{n+\frac{1}{2}}(\lambda_j \rho)}{\sqrt{\rho} J_{n+\frac{1}{2}}(\lambda_j R)},$$

$J_{n+\frac{1}{2}}(\rho)$ is the Bessel function.

Taking into account (13), we can write the particular solution of equation $\Delta h_0 = h$ in the following form

$$h_0(\mathbf{x}) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\rho^2}{3+2n} \left(\frac{\rho}{R}\right)^n Z_n(\xi, \eta). \tag{14}$$

From (12), passing to the limit as $\rho \to R$, and bearing in mind (7), for the determination of the unknown functions we arrive at the following system of algebraic equations

$$\begin{aligned}
&\Omega^+ + 2m_0 h_0^+ - R \left[\frac{\partial}{\partial \rho} \left[(m_0 - 1) h_0 + \sum_{j=1}^{2} \frac{h_j}{\lambda_j^2} \right] \right]_{\rho=R} = q_1^+, \\
&Z_{2n} = q_{5n}^+, \\
&B_0 h^+ + \sum_{j=1}^{2} B_j h_j^+ = q_2^+, \quad C_0 h^+ + \sum_{j=1}^{2} C_j h_j^+ = q_3^+, \\
&h^+ + \sum_{j=1}^{2} h_j^+ = q_4^+.
\end{aligned} \tag{15}$$

Obviously, in view of Theorem 1 we conclude that the determinant of system (15) is different from zero. Therefore, system (15) is uniquely solvable.We find

$$\begin{aligned}
h^+ &= \frac{1}{\mu m_0} [b q_2^+ + d q_3^+ + \mu_0 q_4^+] = G^+, \\
h_j^+ &= \frac{(-1)^j}{d} \left[\frac{B_1 B_2}{B_j} (q_4^+ - G^+) - q_2^+ + B_0 G^+ \right] = G_j^+.
\end{aligned} \tag{16}$$

where

$$d = B_1 - B_2 = -\frac{(\lambda_1^2 - \lambda_2^2) K_0 d \beta_0}{\mu_0 \delta_1 \delta_2} = \frac{(\lambda_1^2 - \lambda_2^2) d \alpha_0 \mu_0}{K_0 \beta_0},$$

$$\delta_1 \delta_2 = -\frac{\beta_0^2 K_0^2}{\mu_0^2 \alpha_0}, \quad K_0 = B_0(b\beta - \alpha d) + C_0(\gamma b - d\beta),$$

On the other hand, as seen from relations (13) and (16),for determining unknown functions, we get

$$Z_n = G_n^+, \quad Y_{jn} = G_{jn}^+, \quad j = 1, 2,$$

where

$$G_n^+ = \frac{2n+1}{4\pi R^2} \iint\limits_S P_n(\cos\gamma) G^+(\mathbf{y}) dS_y,$$

$$G_{jn}^+ = \frac{2n+1}{4\pi R^2} \iint\limits_S P_n(\cos\gamma) G_j^+(\mathbf{y}) dS_y.$$

Since the functions h, h_0 and h_j are already determined, let us calculate the function Ω^+ from equation $(15)_1$. We get

$$\Omega^+ = -2m_0 h_0^+ + R\left[\frac{\partial}{\partial\rho}\left[(m_0 - 1)h_0 + \sum_{j=1}^{2} \frac{h_j}{\lambda_j^2}\right]\right]_{\rho=R} + q_1^+ = G_3^+.$$

For determining unknown function Z_{1n}, from (13) we get

$$Z_{1n} = G_{3n}^+,$$

where

$$G_{3n}^+ = \frac{2n+1}{4\pi R^2} \iint\limits_S P_n(\cos\gamma) G_3^+(\mathbf{y}) dS_y.$$

This finally leads that the functions h, h_j, Ω, h_0, $\sum\limits_{k=1}^{3} \dfrac{\partial \Psi_k}{\partial S_k(\mathbf{z})}$ are defined as

$$\begin{aligned}
h(\mathbf{x}) &= \sum_{n=0}^{\infty} \frac{\rho^n}{R^n} G_n^+(\xi,\eta) = \frac{1}{4\pi R} \iint\limits_S \frac{R^2-\rho^2}{r^3(\mathbf{x},\mathbf{y})} G^+(\mathbf{y}) ds_y,\\
h_j(\mathbf{x}) &= \sum_{n=0}^{\infty} \phi_n(\lambda_j\rho) G_{jn}^+(\xi,\eta), \quad \rho < R,\\
\Omega(\mathbf{x}) &= \sum_{n=0}^{\infty} \frac{\rho^n}{R^n} G_{3n}^+(\xi,\eta) = \frac{1}{4\pi R} \iint\limits_S \frac{R^2-\rho^2}{r^3(\mathbf{x},\mathbf{y})} G_3^+(\mathbf{y}) ds_y,\\
\sum_{k=1}^{3} \frac{\partial \Psi_k}{\partial S_k(\mathbf{z})} &= \sum_{n=0}^{\infty} \frac{\rho^n}{R^n} q_{5n}^+(\xi,\eta) = \frac{1}{4\pi R} \iint\limits_S \frac{R^2-\rho^2}{r^3(\mathbf{x},\mathbf{y})} q_5^+(\mathbf{y}) ds_y,
\end{aligned} \tag{17}$$

$$h_0(\mathbf{x}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{\rho^2}{3+2n}\left(\frac{\rho}{R}\right)^n G_n^+(\xi,\eta), \quad r(\mathbf{x},\mathbf{y}) = \sum_{n=0}^{3}(x_j - y_j)^2.$$

Substituting (17) into (4), we get the solution of Problem 1.

For absolutely and uniformly convergence of obtained series, together with their first derivatives, it is sufficient to assume that

$$f_j \in C^5(S), \quad j = 1, 2, .., 5.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

5 The Solution of Problem 2

In this section we can construct a solution of Problem 2 for an elastic porous space with spherical cavity.

On the basis of the identities (6), from (9), after some lengthy calculations, we obtain

$$\begin{aligned}
&(\mathbf{x}\cdot\mathbf{Pu}) = 2\mu\left(\mathbf{x}\cdot\frac{\partial\mathbf{\Psi}}{\partial n}\right) - 2\mu\rho\frac{\partial^2}{\partial\rho^2}\left[(m_0-1)h_0 + \sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right]\\
&+\rho[\lambda\mathrm{div}\mathbf{u} + b\varphi + d\psi],\\
&\sum_{k=1}^{3}\frac{\partial(\mathbf{Pu})_k}{\partial S_k} = \mu\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k},
\end{aligned} \tag{18}$$

$$\mathrm{div}\mathbf{Pu} = \mu m_0\frac{h}{\rho} - \frac{2\mu}{\rho^2}\left(\mathbf{x}\cdot\frac{\partial\mathbf{\Psi}}{\partial n}\right) + \frac{2\mu}{\rho}\frac{\partial^2}{\partial\rho^2}\left[(m_0-1)h_0 + \sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right],$$

$$\left(\mathbf{x}\cdot\frac{\partial\mathbf{\Psi}}{\partial n}\right) = \left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)(\mathbf{x}\cdot\mathbf{\Psi}),$$

where

$$(\mathbf{x}\cdot\mathbf{\Psi}) = \Omega + 2m_0h_0, \tag{19}$$

Ω is an arbitrary harmonic function $\Delta\Omega = 0$ and the function h_0 is a bi-harmonic function and chosen such that $\Delta h_0 = h$.

From (4) and(18) we obtain the following relations

$$\begin{aligned}
&\operatorname{div}\mathbf{Pu} = \frac{\mu m_0 h}{\rho} - \frac{2\mu}{\rho^2}\left(\mathbf{x}\cdot\frac{\partial\mathbf{\Psi}}{\partial n}\right)\\
&+\frac{2\mu}{\rho}\frac{\partial^2}{\partial\rho^2}\left[(m_0-1)h_0+\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right],\\
&\operatorname{div}\mathbf{Pu}+\frac{1}{\rho^2}(\mathbf{x}\cdot\mathbf{Pu}) = \frac{2\mu}{\rho}\left[(m_0-1)h-\sum_{j=1}^{2}h_j\right],
\end{aligned} \tag{20}$$

$$\frac{\partial\varphi}{\partial n} = B_0\frac{\partial h}{\partial n}+\sum_{j=1}^{2}B_j\frac{\partial h_j}{\partial n}, \quad \frac{\partial\psi}{\partial n} = C_0\frac{\partial h}{\partial n}+\sum_{j=1}^{2}C_j\frac{\partial h_j}{\partial n},$$

$$\sum_{k=1}^{3}\frac{\partial(\mathbf{Pu})_k}{\partial S_k} = \mu\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial\Psi_k}{\partial S_k}.$$

We introduce the following functions:

$$(\mathbf{z}\cdot\mathbf{F})^- = q_1^-(\mathbf{z}), \quad \operatorname{div}\mathbf{F} = q_2^-(\mathbf{z}), \quad \sum_{k=1}^{3}\frac{\partial(\mathbf{Pu})_k}{\partial S_k} = q_5^-. \tag{21}$$

Let us assume that functions q_k, $k=1,2$, and f_k, $k=4,5$ are representable in the form of the series

$$q_k^-(\mathbf{y}) = \sum_{n=0}^{\infty} q_{kn}^-(\xi_0,\eta_0), \quad f_k^-(\mathbf{y}) = \sum_{n=0}^{\infty} f_{kn}^-(\xi_0,\eta_0), \tag{22}$$

where q_{kn}^-, $(k=1,2)$, f_{kn}^-, $k=4,5$ are the spherical harmonics of order n

$$q_{kn}^- = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma)q_k^-(\mathbf{y})dS_y, \quad f_{kn}^- = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma)f_k^-(\mathbf{y})dS_y.$$

From (20), passing to the limit as $\rho\to R$, for determining the unknown values, we obtain the following system of algebraic equations;

when $\rho = R$

$$\begin{aligned}
&\left(\frac{\partial}{\partial\rho}-\frac{1}{R}\right)(\Omega+2m_0h_0) = \frac{R^2}{2\mu}\\
&\left(-q_2^-+\frac{\mu m_0 h^-}{R}+\frac{2\mu}{R}\frac{\partial^2}{\partial\rho^2}\left[(m_0-1)h_0+\sum_{j=1}^{2}\frac{h_j}{\lambda_j^2}\right]\right)_{\rho=R} = G_3^-,
\end{aligned}$$

$$\begin{aligned}
&(m_0-1)\left(\frac{\partial h}{\partial \rho}\right)^- - \sum_{j=1}^{2}\left(\frac{\partial h_j}{\partial \rho}\right)^- \\
&=\left(\frac{R}{2\mu}\frac{\partial}{\partial \rho}\left[\mathrm{div}\mathbf{Pu}+\frac{1}{\rho^2}(\mathbf{x}\cdot\mathbf{Pu})\right]\right)_{\rho=R}=\omega^-, \\
&B_0\left(\frac{\partial h}{\partial \rho}\right)^- + \sum_{j=1}^{2}B_j\left(\frac{\partial h_j}{\partial \rho}\right)^- = f_4^-, \\
&C_0\left(\frac{\partial h}{\partial \rho}\right)^- + \sum_{j=1}^{2}C_j\left(\frac{\partial h_j}{\partial \rho}\right)^- = f_5^-, \\
&\mu\left(\frac{\partial}{\partial \rho}-\frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial \Psi_k}{\partial S_k}=q_5^-.
\end{aligned} \tag{23}$$

On the basis of Theorem 1, we conclude that the determinant of system (23) is different from zero and the system (23) is always solvable. After some lengthy calculations, from (23) we get

$$\begin{aligned}
&\left(\frac{\partial h}{\partial \rho}\right)^- = \frac{1}{(\lambda+\mu)m_0}\left[\mu_0\omega^- - b\, f_4^- - d\, f_5^-\right]=G^-, \\
&\left(\frac{\partial h_j}{\partial \rho}\right)^- = \frac{(-1)^j}{d}\left[B_0G^- - f_4^- + \frac{B_1B_2}{B_j}\left((m_0-1)G^- - \omega^-\right)\right] \\
&=G_j^-,
\end{aligned} \tag{24}$$

where

$$d=B_1-B_2=\frac{(\lambda_1^2-\lambda_2^2)d\alpha_0\mu_0}{K_0\beta_0},$$

$$K_0=B_0(b\beta-\alpha d)+C_0(\gamma b-d\beta).$$

Let us assume that the functions h, Ω and h_j, $(j=1,2)$ in (23) are sought in the form [34]

$$\begin{aligned}
&\Omega=-\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{R^{n+2}}{\rho^{n+1}}Y_n(\xi,\eta), \\
&h(\mathbf{x})=-\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{R^{n+2}}{\rho^{n+1}}Z_n(\xi,\eta), \\
&\sum_{k=1}^{3}\frac{\partial \Psi_k}{\partial S_k}=-\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{R^{n+2}}{\rho^{n+1}}Z_{3n}(\xi,\eta), \\
&h_j(\mathbf{x})=\sum_{n=0}^{\infty}\Psi_n(\lambda_j\rho)Z_{jn}(\xi,\eta) \qquad \rho>R.
\end{aligned} \tag{25}$$

where Y_n, Z_n and Z_{jn}, $(j = 1, 2, 3)$ are the unknown spherical harmonic of order n,

$$\Psi_m(\lambda_k\rho) = \frac{\sqrt{R}H^{(1)}_{m+\frac{1}{2}}(\lambda_k\rho)}{\sqrt{\rho}H^{(1)}_{m+\frac{1}{2}}(\lambda_k R)},$$

$H^{(1)}_{m+\frac{1}{2}}(z)$ is the Hankel's function.

On the basis of equation $\Delta h_0 = h$, the function h_0 can be represented in the following form

$$h_0(\mathbf{x}) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\rho^2}{1-2n}\frac{R^{n+2}}{\rho^{n+1}} Z_n(\xi, \eta). \tag{26}$$

On the other hand, substitution of (25) into (24) yields

$$Z^-_n = G^-_n, \quad Z_{jn} = \frac{G^-_{jn}}{\left(\frac{\partial}{\partial\rho}\Psi_n(\lambda_j\rho)\right)_{\rho=R}}, \quad Z_{3n} = \frac{q^-_{5n}}{2\mu}, \tag{27}$$

where

$$G^-_n = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma)G^-(\mathbf{y})dS_y,$$

$$G^-_{jn} = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma)G^-_j(\mathbf{y})dS_y.$$

and (25)-(26) takes the form

$$\begin{aligned}
&h(\mathbf{x}) = -\sum_{n=0}^{\infty} \frac{1}{n+1}\frac{R^{n+2}}{\rho^{n+1}} G^-_n(\xi, \eta),\\
&h_j(\mathbf{x}) = \sum_{n=0}^{\infty} \Psi_n(\lambda_j\rho)\frac{G^-_{jn}}{\left(\frac{\partial}{\partial\rho}\Psi_n(\lambda_j\rho)\right)_{\rho=R}}, \qquad \rho > R,\\
&h_0(\mathbf{x}) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\rho^2}{1-2n}\frac{R^{n+2}}{\rho^{n+1}} G^-_n(\xi, \eta),\\
&\sum_{k=1}^{3} \frac{\partial\Psi_k}{\partial S_k} = -\frac{1}{2\mu}\sum_{n=0}^{\infty} \frac{1}{n+1}\frac{R^{n+2}}{\rho^{n+1}} q^-_{5n},
\end{aligned} \tag{28}$$

where G^-_n and G^-_{jn} are known spherical harmonics.

Since the functions h, h_0 and h_j having already determined, to determination of Ω, from $(23)_1$ we get the following BVP:

find the harmonic function Ω *in* D^-, *with the boundary condition on* S

$$\left(\frac{\partial}{\partial n}-\frac{1}{R}\right)\Omega=G_3^- - 2m_0\left[\left(\frac{\partial}{\partial \rho}-\frac{1}{R}\right)h_0\right]_{\rho=R}=G_4,\quad \rho=R.$$

Thus, for the Laplace equation we have obtained the Robin boundary value problem, the solution of which is

$$\Omega(\mathbf{x})=-\sum_{n=0}^{\infty}\frac{1}{n+2}\frac{R^{n+2}}{\rho^{n+1}}G_{4n}^-(\xi,\eta),\quad \rho>R,$$

where

$$G_{4n}^-=G_{3n}^-+\frac{m_0R^2n}{1-2n}G_n^-.$$

For absolutely and uniformly convergence of obtained series, together with their first derivatives, it is sufficient to assume that

$$f_j\in C^5(S),\quad j=1,2,..,5.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

Thus, the considered problems have been solved completely.

We note that in the elasticity theory of isotropic bodies, the basic BVPs for the sphere in the classical setting for potential methods are thoroughly investigated in [33] (see also references therein).

6 Conclusions

In this paper the following result is obtained: the Dirichlet type and the Neumann type BVPs for the sphere with double voids and for a space with spherical cavity are solved explicitly.For the harmonic functions the Poisson type formulas are obtained. The bi-harmonic and meta-harmonic functions are presented as absolutely and uniformly convergent series.

References

1. Biot MA. General theory of three dimensional consolidation. *J. of Applied Physics,* **12** (1941), 155- 164.

2. Biot, MA. Mechanics of deformation and acoustic propagation in porous media. *J. of Applied Physics,* **33** (1962), 1482-1498.

3. Nunziato G.W. and Cowin S.C. A nonlinear theory of elastic materials with voids. *Arch.Ration. Mech.Anal.,* **72** (1979), 175-201.

4. Cowin S.C. and Nunziato G.W. Linear elastic materials with voids. *J. Elasticity,* **13** (1983), 125-147.

5. Ieşan D. and Quintanilla R. On a theory of thermoelastic materials with a double porosity structure. *J. Thermal Stresses*, **37** (2014), 1017-1036.

6. Ieşan D. A Theory of Thermoelastic Materials with voids, *Acta Mechanica*, **60** (1986), 67-89.

7. de Boer R. Theory of porous media,Highlights in the Historical Development and Current State. *Springer,Berlin* (2000).

8. Straughan B. Stability and Wave Motion in porous media. *Springer, New York* (2008).

9. Straughan B.Convection with Local Thermal Non-equilibrium and Microfluidic Effect. *Springer,Berlin* (2015).

10. Svanadze M.Potential method in mathematical theories of multi-porosity media. *Interdisciplinary Applied Mathematics.* **51** *Springer, Switzerland.* 2019.

11. Svanadze M. Potential method in the coupled theory of elastic double porosity materials. *J. Acta Mech.*, (2021), doi.org/101007/s00707-020-02921-2.

12. Svanadze M. On a theory of thermoelastic materials with a double porosity structure. *Thermal Stresses,* **17** (2014), 1017-1036.

13. Khalili N. and Selvadurai P. S. A fully coupled constitutive model for thermo-hydro-mechanical analysis in elastic media with double porosity, *Geophysical Research Letters*, **30** (2003), pp.SDE 7-1-7-3.

14. Straughan B. Stability and uniqueness in double porosity elasticity. *Int.J. of Engineering Science*, **65** (2013), 1-8.

15. Ciarletta M., Scalia A. On uniqueness and reciprocity in linear thermoelasticity of materials with voids. *J. Elasticity* **32** (1993), 1-17.

16. Ciarletta M., Scalia A. Results and applications in thermoelasticity of materials with voids. *Le Matematiche*, **XLVI** (1991), 85-96.

17. Singh J., Tomar S. K. Plane waves in thermo-elastic materials with voids. *Mechanics of Materials* **39** (2007), 932-940.

18. Singh J. Wave propagation in a generalized thermoelastic material with voids. *Applied Mathematics and Computation*, **189** (2007), 698-709.

19. Puri P., Cowin S. C. Plane waves in linear elastic materials with voids. *J.Elasticity* **15** (1985), 167-183.

20. Ieşan D., Nappa L. Axially symmetric problems for porous elastic solid. *Int.J.Solid Struct.*, **40** (2003), 5271-5286.

21. Chirita S. and Scalia A. On the spatial and temporal behavior in linear thermoelasticity of material with voids. *J. Thermal Stresses,* **24** (2001), 433-455.

22. Ciarletta M., Svanadze M. and Buonanno L. Plane waves and vibrations in the theory of micropolar thermoelasticity for material with voids. *European J. of Mechanics-A-Solids,* **28** (2009), 897-903.

23. Bitsadze L., Tsagareli I. Solutions of BVPs in the fully coupled theory of elasticity for the space with double porosity and spherical cavity. *Mathematical Methods in the Applied Science.* **39** (2016), 2136-2145.

24. Bitsadze L., Tsagareli I. The solution of the Dirichlet BVP in the fully coupled theory for spherical layer with double porosity. *Meccanica*, **51** (2016), 1457-1463.

25. Bitsadze L. Explicit solution of the Dirichlet boundary value problem of elasticity for porous infinite strip, (ZAMP,Journal of Applied Mathematics and Physics), Zeitschrift fr angewandte Mathematik und Physik, **71** (5), Article number: 145 (2020), DOI: 10.1007/s00033-020-01379-5

26. Bitsadze L. Explicit solutions of the BVPs of the theory of thermoelasticity for an elastic circle with voids and microtemperatures(ZAMM). *Journal of Applied Mathematics and Mechanics*, **100** (2020), https://doi.org/10.1002/zamm.201800303.

27. Bitsadze L. Boumdary value problems for an infinite strip with voids. *Proceedings of I. Vekua Institute of Applied Mathematics.* **69** (2019), 13-23.

28. Bitsadze L. Explicit solutions of quasi-static problems in the coupled theory of porooelasticity. *Continuum Mechanics and Thermodynamics*, https: doi.org/10.1007/s00161-021-01029-9

29. Tsagareli I., Bitsadze L. Explicit solution of one boundary value problem in the full coupled theory of elasticity for solids with double porosity. *Acta Mech .* **226**, 5 (2015), 1409-1418.

30. Bitsadze L. Explicit solutions of boundary value problems of elasticity for circle with a double voids structure. *J. of the Brazilian*

Society of Mechanical Sciences and Engineering **41**, 9 (2019), DOI: 10.1007/s40430-019-1888-3.

31. Vashakmadze T. Some mathematical problems of poroelasticity: modeling, analysis, design, and its applications. *Bulletin of TICMI* **6** (2002), 27-30.

32. Natroshvili D.G., Svanadze M.G. Some dynamical problems of the theory of coupled thermoelasticity for the piecewise homogeneous bodies, *Proceedings of I. Vekua Institute of Applied Mathematics*, **10** (1981), 99-190, Tbilisi.

33. Kupradze V.D., Gegelia T.G., Basheleishvili M.O., Burchuladze T.V. *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity.* North-Holland Publ. Company, Amsterdam-New-York- Oxford, 1979.

34. Smirnov V.I. Course of Higher Mathematics. vol. III, part 2, Moscow: Nauka, 1969.