

ON REALIZATION OF A NUMERICAL ALGORITHM FOR A TIMOSHENKO BEAM

N. Kachakhidze¹, J. Peradze², Z. Tsiklauri³

¹ Department of Mathematics, Georgian Technical University
77, M. Kostava Str., Tbilisi 0175, Georgia
n.kachakhidze@gtu.ge

² Department of Mathematics, I. Javakishvili Tbilisi State University
2, University Str., Tbilisi 0186, Georgia
j_peradze@yahoo.com

³ Department of Mathematics, Georgian Technical University
77, M. Kostava Str., Tbilisi 0175, Georgia
zviad.tsiklauri@yahoo.com

(Received 23.01.2020; accepted 06.06.2020)

Abstract

An initial boundary value problem is considered for the nonlinear integro-differential system, which describes the dynamic behavior of the beam. The solution is approximated by the finite element method, an implicit difference scheme and a Picard type iteration method. The algorithm has been checked by the tests. The results of calculations are given.

Keywords and phrases: Timoshenko beam model, finite element method, difference scheme, iteration method, realization of a numerical algorithm.

AMS subject classification (2010): 35L20, 74H15, 74S05, 74S20, 65H10.

1 Statement of the Problem

We consider the problem of geometrically nonlinear vibration of a beam. To this end, we use the well known model of Timoshenko which is the second generation theory. As compared with the classical Kirchhoff-Love model, in the Timoshenko model, deformation due to cross force and rotary inertia is taken into account, which is rather important for many problems.

Let us consider the system of equations

$$\begin{aligned}w_{tt} &= \left(cd - a + b \int_0^1 w_x^2 dx \right) w_{xx} - cd\psi_x + \alpha(x, t), \\ \psi_{tt} &= c\psi_{xx} - c^2 d(\psi - w_x) + \beta(x, t), \\ 0 &< x < 1, \quad 0 < t \leq T,\end{aligned}\tag{1}$$

with the initial and boundary conditions

$$\begin{aligned} w_t(x, 0) &= w^{(1)}(x), & w(x, 0) &= w^{(2)}(x), \\ \psi_t(x, 0) &= \psi^{(1)}(x), & \psi(x, 0) &= \psi^{(2)}(x), \\ w_t(0, t) = w_t(1, t) &= 0, & \psi_t(0, t) = \psi_t(1, t) &= 0, \\ 0 \leq x \leq 1, & & 0 \leq t \leq T. \end{aligned} \quad (2)$$

The unknown functions w and ψ are respectively the transverse deflection of the beam centerline and the rotary displacement of the cross-section, $\alpha(x, t)$ and $\beta(x, t)$ are the given functions of the force components.

In (2) $w^{(l)}, \psi^{(l)}$ are the given functions,

$$w^{(l)}(x) \in W_2^2(0, 1), \quad \psi^{(l)}(x) \in W_2^{l+1}(0, 1), \quad l = 1, 2.$$

In (1)

$$a, b, c, d > 0, \quad cd - a > 0, \quad (3)$$

where

$$a = Al\Delta/I_1, \quad b = Al^2/(2I_1), \quad c = Al^2/I_2, \quad d = GI_2/(EI_1), \quad (4)$$

E is Young's modulus, G is the shear modulus, A is the cross-section area, l is the length, I_1 is the moment of inertia of the cross-section about the axis perpendicular to the beam centerline, I_2 is the polar moment of inertia of the cross-section, and Δ is the end shortening of the beam.

By virtue of (4) we come to a conclusion that the second relation in (3) is a natural requirement for moderately compressed slender beams, which is equivalent to the condition $\Delta < lG/E$.

System (1) was obtained in [9] when describing a dynamic beam using Timoshenko theory [10]. The solvability of this system is investigated in [11] and [6]. Our goal consists in constructing an approximate algorithm of the solution of problem (1), (2) and describing results of some computational testing. Numerical methods for system (1) and other type Timoshenko beam models are investigated in [1], [2], [3], [4], [5], [6], [7], [8], [12].

2 Reducing (1) to a system of first order equations

Let us write a set of functions

$$u, v, f, \varphi, \psi, \quad (5)$$

of which the first four are the new ones defined by the formulas

$$u = w_t, \quad v = w_x, \quad f = \psi_t, \quad \varphi = \psi_x. \quad (6)$$

Using (1) and (6), we write a system of equations with respect to functions (5) as

$$\begin{aligned} u_t &= \left(cd - a + b \int_0^1 v^2 dx \right) v_x - cd\varphi + \alpha(x, t), \\ v_t &= u_x, \quad f_t = c\varphi_x - c^2 d(\psi - v) + \beta(x, t), \\ \varphi_t &= f_x, \quad \psi_t = f, \\ 0 &< x < 1, \quad 0 < t \leq T. \end{aligned} \quad (7)$$

Let us complement (7) with the initial and boundary conditions which follow from (2)

$$\begin{aligned} u(x, 0) &= w^{(1)}(x), \quad v(x, 0) = w_x^{(2)}(x), \quad f(x, 0) = \psi^{(1)}(x), \\ \varphi(x, 0) &= \psi_x^{(2)}(x), \quad \psi(x, 0) = \psi^{(2)}(x), \\ u(0, t) &= u(1, t) = 0, \quad f(0, t) = f(1, t) = 0, \\ 0 &\leq x \leq 1, \quad 0 \leq t \leq T. \end{aligned} \quad (8)$$

Thus, instead of (1), (2), we have obtained the initial-boundary problem (7), (8) for functions (5) which will be considered below.

3 Algorithm

1. Finite element method

We discretize problem (7), (8) with respect to the spatial variable by means of the finite element method.

Assume that the interval $[0, 1]$ is covered by a net with the step $h = 1/N$, where N is a positive integer number. To each node $x_i = ih$, $i = 0, 1, \dots, N$, we assign the base function from the set

$$\omega_{hi}(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in (x_{i-1}, x_i), \\ \frac{x_{i+1} - x}{h}, & x \in (x_i, x_{i+1}), \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases} \quad i = 1, 2, \dots, N-1,$$

$$\omega_{h0}(x) = \begin{cases} \frac{x_1 - x}{h}, & x \in (x_0, x_1), \\ 0, & x \notin (x_0, x_1), \end{cases}$$

$$\omega_{hN}(x) = \begin{cases} \frac{x - x_{N-1}}{h}, & x \in (x_{N-1}, x_N), \\ 0, & x \notin (x_{N-1}, x_N). \end{cases}$$

Denote by (\cdot, \cdot) the scalar product in $L^2(0, 1)$. An approximate solution of problem (7), (8) will be sought in the form

$$u_h(x, t) = \sum_{i=1}^{N-1} u_i(t)\omega_{hi}, \quad v_h(x, t) = \sum_{j=0}^N v_j(t)\omega_{hj},$$

$$f_h(x, t) = \sum_{i=1}^{N-1} f_i(t)\omega_{hi}, \quad \varphi_h(x, t) = \sum_{j=0}^N \varphi_j(t)\omega_{hj}, \quad \psi_h(x, t) = \sum_{j=0}^N \psi_j(t)\omega_{hj}.$$

Besides, we will use the following approximations of functions $\alpha(x, t)$ and $\beta(x, t)$

$$\alpha_h(x, t) = \sum_{j=0}^N \alpha_j(t)\omega_{hj}, \quad \beta_h(x, t) = \sum_{j=0}^N \beta_j(t)\omega_{hj},$$

where $\alpha_j(t) = \alpha(x_j, t)$, $\beta_j(t) = \beta(x_j, t)$.

The functions $u_i(t)$, $v_j(t)$, $f_i(t)$, $\varphi_j(t)$, $\psi_j(t)$ are defined by the system of ordinary differential equations

$$(u_{ht}, \omega_{hi}) = ((cd - a + b \int_0^1 v_h^2 dx)v_{hx} - cd\varphi_h + \alpha_h(x, t), \omega_{hi}),$$

$$(v_{ht}, \omega_{hj}) = (u_{hx}, \omega_{hj}), \quad (f_{ht}, \omega_{hi}) = (c\varphi_{hx} - c^2d(\psi_h - v_h) + \beta_h(x, t), \omega_{hi}),$$

$$(\varphi_{ht}, \omega_{hj}) = (f_{hx}, \omega_{hj}), \quad (\psi_{ht}, \omega_{hj}) = (f_h, \omega_{hj}), \tag{9}$$

$0 < t \leq T$, with the initial conditions

$$u_i(0) = w^{(1)}(x_i), \quad v_j(0) = w_x^{(2)}(x_j), \quad f_i(0) = \psi^{(1)}(x_i),$$

$$\varphi_j(0) = \psi_x^{(2)}(x_j), \quad \psi_j(0) = \psi^{(2)}(x_j). \tag{10}$$

In (9), (10), $i = 1, 2, \dots, N - 1$, $j = 0, 1, \dots, N$.

Let us introduce some notation. To functions λ_h and μ_h of the form

$$\lambda_h = \sum_{i=1}^{N-1} \lambda_i(t)\omega_{hi}, \quad \mu_h = \sum_{j=0}^N \mu_j(t)\omega_{hj}$$

we assign the vectors

$$\boldsymbol{\lambda}_h(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_{N-1}(t))',$$

$$\boldsymbol{\mu}_h(t) = (\mu_0(t), \mu_1(t), \dots, \mu_N(t))'. \tag{11}$$

Here the symbol $'$ means the operation of transformation.

Now (9), (10) can be rewritten as system

$$\begin{aligned}
 M \frac{d\mathbf{u}_h}{dt} &= (cd - a + bh\mathbf{v}'_h K \mathbf{v}_h) Q \mathbf{v}_h - cdL\boldsymbol{\varphi}_h + L\boldsymbol{\alpha}_h, \\
 K \frac{d\mathbf{v}_h}{dt} &= -Q' \mathbf{u}_h, \\
 M \frac{d\mathbf{f}_h}{dt} &= cQ\boldsymbol{\varphi}_h - c^2 dL(\boldsymbol{\psi}_h - \mathbf{v}_h) + L\boldsymbol{\beta}_h, \\
 K \frac{d\boldsymbol{\varphi}_h}{dt} &= -Q' \mathbf{f}_h, \\
 K \frac{d\boldsymbol{\psi}_h}{dt} &= L' \mathbf{f}_h,
 \end{aligned} \tag{12}$$

$0 < t \leq T$, with the initial conditions

$$\begin{aligned}
 \mathbf{u}_h(0) &= \mathbf{w}^{(1)}, \quad \mathbf{v}_h(0) = \mathbf{w}_x^{(2)}, \quad \mathbf{f}_h(0) = \boldsymbol{\psi}^{(1)}, \\
 \boldsymbol{\varphi}_h(0) &= \boldsymbol{\psi}_x^{(2)}, \quad \boldsymbol{\psi}_h(0) = \boldsymbol{\psi}^{(2)}.
 \end{aligned} \tag{13}$$

In (12) and (13)

$$\begin{aligned}
 K &= \frac{1}{h}((\omega_{hi}, \omega_{hj}))_{0 \leq i, j \leq N} = \frac{1}{6} \begin{pmatrix} 2 & 1 & & & & \\ 1 & 4 & & & & 0 \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & 4 & 1 \\ & & & & & 1 & 2 \end{pmatrix}, \\
 L &= \frac{1}{h}((\omega_{hi}, \omega_{hj}))_{\substack{1 \leq i \leq N-1 \\ 0 \leq j \leq N}} = \frac{1}{6} \begin{pmatrix} 1 & 4 & 1 & & & & \\ & & \cdot & & & & 0 \\ & & & \cdot & & & \\ 0 & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 & 4 & 1 \end{pmatrix}, \\
 M &= \frac{1}{h}((\omega_{hi}, \omega_{hj}))_{1 \leq i, j \leq N-1} = \frac{1}{6} \begin{pmatrix} 4 & 1 & & & & \\ 1 & \cdot & & & & 0 \\ & & \cdot & & & \\ 0 & & & \cdot & & 1 \\ & & & & \cdot & 4 \end{pmatrix}, \\
 Q &= -\frac{1}{h}((\omega_{hix}, \omega_{hj}))_{\substack{1 \leq i \leq N-1 \\ 0 \leq j \leq N}} = \frac{1}{2h} \begin{pmatrix} -1 & 0 & 1 & & & & \\ & & \cdot & & & & 0 \\ & & & \cdot & & & \\ 0 & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & -1 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

and

$$\begin{aligned} \mathbf{w}^{(1)} &= (w^{(1)}(x_1), w^{(1)}(x_2), \dots, w^{(1)}(x_{N-1}))', \\ \mathbf{w}_x^{(2)} &= (w_x^{(2)}(x_0), w_x^{(2)}(x_1), \dots, w_x^{(2)}(x_N))', \\ \boldsymbol{\psi}^{(l)} &= (\psi^{(l)}(x_{2-l}), \psi^{(l)}(x_{3-l}), \dots, \psi^{(l)}(x_{l+N-2}))', \quad l = 1, 2, \\ \boldsymbol{\psi}_x^{(2)} &= (\psi_x^{(2)}(x_0), \psi_x^{(2)}(x_1), \dots, \psi_x^{(2)}(x_N))'. \end{aligned}$$

2. Difference scheme

Let us derive an approximate solution of problem (12), (13). On the time interval $[0, T]$ we introduce a net with the step $\tau = \frac{T}{P}$ and nodes $t_n = n\tau$, $n = 0, 1, \dots, P$. On the n -th layer, i.e. for $t = t_n$, the approximate values of vectors (11) are denoted by $\boldsymbol{\lambda}_h^n$ and $\boldsymbol{\mu}_h^n$, $n = 0, 1, \dots, P$. We use the modified Crank–Nicolson type scheme [6]

$$\begin{aligned} M(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) &= \frac{\tau}{4} \{2(cd - a) + bh[(\mathbf{v}_h^n)'K\mathbf{v}_h^n + (\mathbf{v}_h^{n-1})'K\mathbf{v}_h^{n-1}]\} Q(\mathbf{v}_h^n + \mathbf{v}_h^{n-1}) \\ &\quad - \frac{\tau cd}{2} L(\boldsymbol{\varphi}_h^n + \boldsymbol{\varphi}_h^{n-1}) + \frac{\tau}{2} L(\boldsymbol{\alpha}_h^n + \boldsymbol{\alpha}_h^{n-1}), \\ 2K(\mathbf{v}_h^n - \mathbf{v}_h^{n-1}) &= -\tau Q'(\mathbf{u}_h^n + \mathbf{u}_h^{n-1}), \\ M(\mathbf{f}_h^n - \mathbf{f}_h^{n-1}) &= \frac{\tau c}{2} Q(\boldsymbol{\varphi}_h^n + \boldsymbol{\varphi}_h^{n-1}) \\ &\quad - \frac{\tau c^2 d}{2} L(\boldsymbol{\psi}_h^n + \boldsymbol{\psi}_h^{n-1} - \mathbf{v}_h^n - \mathbf{v}_h^{n-1}) + \frac{\tau}{2} L(\boldsymbol{\beta}_h^n + \boldsymbol{\beta}_h^{n-1}), \\ 2K(\boldsymbol{\varphi}_h^n - \boldsymbol{\varphi}_h^{n-1}) &= -\tau Q'(\mathbf{f}_h^n + \mathbf{f}_h^{n-1}), \\ 2K(\boldsymbol{\psi}_h^n - \boldsymbol{\psi}_h^{n-1}) &= \tau L'(\mathbf{f}_h^n + \mathbf{f}_h^{n-1}), \end{aligned} \tag{14}$$

$n = 1, 2, \dots, P$, with the initial conditions

$$\mathbf{u}_h^0 = \mathbf{w}^{(1)}, \quad \mathbf{v}_h^0 = \mathbf{w}_x^{(2)}, \quad \mathbf{f}_h^0 = \boldsymbol{\psi}^{(1)}, \quad \boldsymbol{\varphi}_h^0 = \boldsymbol{\psi}_x^{(2)}, \quad \boldsymbol{\psi}_h^0 = \boldsymbol{\psi}^{(2)}. \tag{15}$$

We introduce into consideration the vectors $\mathbf{y}_h^n = (\mathbf{u}_h^n, \mathbf{v}_h^n, \mathbf{f}_h^n, \boldsymbol{\varphi}_h^n, \boldsymbol{\psi}_h^n)'$, $\mathbf{g}_h = (\mathbf{w}^{(1)}, \mathbf{w}_x^{(2)}, \boldsymbol{\psi}^{(1)}, \boldsymbol{\psi}_x^{(2)}, \boldsymbol{\psi}^{(2)})'$ and $\boldsymbol{\rho}_h^n = (\boldsymbol{\alpha}_h^n, 0, \boldsymbol{\beta}_h^n, 0, 0)'$, where 0 is the $(N - 1)$ -dimensional zero vector.

System (14) can be rewritten as

$$A \frac{\mathbf{y}_h^n - \mathbf{y}_h^{n-1}}{\tau} = \frac{1}{2} (B + C(\mathbf{v}_h^n) + C(\mathbf{v}_h^{n-1})) (\mathbf{y}_h^n + \mathbf{y}_h^{n-1}) + \frac{1}{2} D(\boldsymbol{\rho}_h^n + \boldsymbol{\rho}_h^{n-1}) \tag{16}$$

and (15) as

$$\mathbf{y}_h^0 = \mathbf{g}_h. \tag{17}$$

In (16) A , B and C are the block square matrices of order five defined by the formulas

$$\begin{aligned}
 A &= \begin{pmatrix} M & & & & \\ & 2K & & 0 & \\ & & M & & \\ & 0 & & 2K & \\ & & & & 2K \end{pmatrix}, \\
 B &= \begin{pmatrix} 0 & (cd - a)Q & 0 & -cdL & 0 \\ -2Q' & 0 & 0 & 0 & 0 \\ 0 & c^2dL & 0 & cQ & -c^2dL \\ 0 & 0 & -2Q' & 0 & 0 \\ 0 & 0 & 2L' & 0 & 0 \end{pmatrix}, \\
 C(\nu) &= \frac{bh}{2}\nu'K\nu \begin{pmatrix} 0 & Q & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \quad \nu \in R^{N+1}, \\
 D &= \begin{pmatrix} L & & & & \\ & 0 & & 0 & \\ & & L & & \\ & 0 & & 0 & \\ & & & & 0 \end{pmatrix}.
 \end{aligned}$$

In A , B and C 0 are the rectangular zero matrices whose dimensions are defined as follows: the number of rows (columns) in the matrix is equal to $N - 1$ if the matrix is in the first or the third row (column) of the block matrix, and to $N + 1$ in all other cases. In D 0 is the rectangular zero matrix whose dimensions are defined as follows: the number of rows (columns) in the matrix is equal to $N - 1$ ($N + 1$) if the matrix is in the first or the third row (column) of the block matrix, and to $N + 1$ ($N - 1$) in all other cases.

3. Iteration process

Let us derive a numerical solution of problem (16), (17). If in this system we perform calculations from layer to layer and assume that \mathbf{y}_h^{n-1} is known, then the problem reduced to finding \mathbf{y}_h^n . For this we use a Picard type iteration process of the form

$$\begin{aligned}
 A\mathbf{y}_h^{n,m} &= A\mathbf{y}_h^{n-1} \\
 &+ \frac{\tau}{2}(B + C(\mathbf{v}_h^{n,m-1}) + C(\mathbf{v}_h^{n-1}))(\mathbf{y}_h^{n,m-1} + \mathbf{y}_h^{n-1}) + \frac{\tau}{2}D(\boldsymbol{\rho}_h^n + \boldsymbol{\rho}_h^{n-1}).
 \end{aligned} \tag{18}$$

Here

$$\mathbf{y}_h^{n,m} = (\mathbf{u}_h^{n,m}, \mathbf{v}_h^{n,m}, \mathbf{f}_h^{n,m}, \boldsymbol{\varphi}_h^{n,m}, \boldsymbol{\psi}_h^{n,m}) \quad (19)$$

is the m -th iteration approximation of the vector \mathbf{y}_h^n and moreover, $\mathbf{u}_h^{n,m}, \mathbf{f}_h^{n,m} \in R^{N-1}$, $\mathbf{v}_h^{n,m}, \boldsymbol{\varphi}_h^{n,m}, \boldsymbol{\psi}_h^{n,m} \in R^{N+1}$. Not to complicate the discussion we assume that \mathbf{y}_h^{n-1} is defined with such an accuracy that the corresponding error can be ignored. Suppose that \mathbf{y}_h^{n-1} is chosen as the initial approximation for iteration process on the upper neighboring n -th layer. Therefore, let in (18) $\mathbf{y}_h^{n,0} = \mathbf{y}_h^{n-1}$.

Remark. The sought functions of the initial problem (1), (2) are the functions $w(x, t)$ and $\psi(x, t)$. The algorithm we have used here allows us to find approximate values of the derivatives w_t, w_x, ψ_t, ψ_x and of the function ψ . Using function $w^{(2)}(x)$ and vector $\mathbf{u}_h^{n,m}$ it is possible to construct an approximation for function $w(x, t)$.

4 Definition of the total error of the algorithm

Let us introduce a scalar product and vector norms. If we assume that λ and μ are vectors of the same dimension, whose l -th components are equal to λ_l and μ_l , then the scalar product $(\lambda, \mu)_h = h \sum_l \lambda_l \mu_l$, where the summation involves all components of λ and μ and the norm $\|\lambda\|_h = (\lambda, \lambda)_h^{\frac{1}{2}}$.

Let $\lambda(x, t)$ and $\mu(x, t)$ be any functions defined on $[0, 1] \times [0, T]$ and let the first of these functions satisfy the boundary condition $\lambda(0, t) = \lambda(1, t) = 0$. Consider the vectors

$$\begin{aligned} \boldsymbol{\lambda}(t) &= (\lambda(x_1, t), \lambda(x_2, t), \dots, \lambda(x_{N-1}, t)), \\ \boldsymbol{\mu}(t) &= (\mu(x_0, t), \mu(x_1, t), \dots, \mu(x_N, t)). \end{aligned}$$

Using this notation and functions (5) we introduce the vector

$$\mathbf{y}(t) = (\mathbf{u}(t), \mathbf{v}(t), \mathbf{f}(t), \boldsymbol{\varphi}(t), \boldsymbol{\psi}(t)). \quad (20)$$

From this definition it follows that the vector $\mathbf{y}(t)$ consists of the values of the exact solution of problem (7), (8) at the nodes x_i of the interval $[0, 1]$.

For chosen h and τ and for $t = t_n$, the vector $\mathbf{y}_h^{n,m}$ (19) is the result of the considered algorithm, the purpose of which is to approximate the vector of exact solution $\mathbf{y}(t)$ (20) at m -th iteration step at the time node $t = t_n$. By virtue of this reasoning we can characterize the total error of the algorithm by the difference $\Delta \mathbf{z}_h^{n,m} = \mathbf{y}(t_n) - \mathbf{y}_h^{n,m}$. Note that by (20) and (19)

$$\|\Delta \mathbf{z}_h^{n,m}\|_h = \left(\|\mathbf{u}(t_n) - \mathbf{u}_h^{n,m}\|_h^2 + \|\mathbf{v}(t_n) - \mathbf{v}_h^{n,m}\|_h^2 \right)^{\frac{1}{2}}$$

$$+ \|\mathbf{f}(t_n) - \mathbf{f}_h^{n,m}\|_h^2 + \|\boldsymbol{\varphi}(t_n) - \boldsymbol{\varphi}_h^{n,m}\|_h^2 + \|\boldsymbol{\psi}(t_n) - \boldsymbol{\psi}_h^{n,m}\|_h^2 \Big)^{\frac{1}{2}}. \quad (21)$$

An estimation of total error (21), when $\alpha(x, t) = \beta(x, t) = 0$ is obtained in [7].

5 Numerical realization

In this section some numerical examples are given.

Example 1. Let

$$\begin{aligned} a &= 0.3, \quad b = 0.2, \quad c = 1.0, \quad d = 0.5, \quad T = 1.0, \\ \alpha(x, t) &= \pi^2 \sin \pi x (0.2 + 0.1\pi^2(t + 1)^4)(t + 1)^2 \\ &\quad + 2 \sin \pi x + 0.5(t - 1)^2 \pi \cos \pi x, \\ \beta(x, t) &= 0.5((t - 1)^2 \sin \pi x - (t + 1)^2 \pi \cos \pi x) + (2 + (t - 1)^2 \pi^2) \sin \pi x, \end{aligned}$$

with the initial functions $w^{(1)}(x) = 2 \sin \pi x$, $w^{(2)}(x) = \pi \cos \pi x$, $\psi^{(1)}(x) = -2 \sin \pi x$, $\psi^{(2)}(x) = \sin \pi x$.

An exact solution and functions (5) have the form $w = (t + 1)^2 \sin \pi x$, $\psi = (t - 1)^2 \sin \pi x$ and $u = 2(t + 1) \sin \pi x$, $v = (t + 1)^2 \pi \cos \pi x$, $f = 2(t - 1) \sin \pi x$, $\varphi = (t - 1)^2 \pi \cos \pi x$, $\psi = (t - 1)^2 \sin \pi x$.

Table 1: Total error of the algorithm

No.	h	τ	n	t_n	m	$\ \Delta \mathbf{z}_h^{n,m}\ _h$
1	0.1	0.01	100	1.0	11	0.0698511
2	0.02	0.002	500	1.0	7	0.0026944
3	0.01	0.001	1000	1.0	6	0.0006710

The values of the norm of the total error (21) of algorithm for some variants are given in Table 1.

Example 2. Let

$$a = 0.3, \quad b = 0.2, \quad c = 1.0, \quad d = 0.5, \quad T = 1.0,$$

and

$$\alpha(x, t) = \frac{2}{5}t + \frac{2}{15}t^3 + \frac{1}{2} \exp(xt)(t(1 - x) - 1),$$

$$\beta(x, t) = \exp(xt) \left((x^2 - t^2 + \frac{1}{2})(1 - x) + 2t \right) - \frac{1}{2}t(1 - 2x),$$

with the initial functions $w^{(1)}(x) = x(1-x)$, $w^{(2)}(x) = 0$, $\psi^{(1)}(x) = x(1-x)$, $\psi^{(2)}(x) = 1 - x$.

An exact solution and functions (5) have the form $w = x(1 - x)t$, $\psi = (1 - x) \exp(xt)$ and $u = x(1 - x)$, $v = t(1 - 2x)$, $f = x(1 - x) \exp(xt)$, $\varphi = (t(1 - x) - 1) \exp(xt)$, $\psi = (1 - x) \exp(xt)$.

Table 2: Total error of the algorithm

No.	h	τ	n	t_n	m	$\ \Delta z_h^{n,m}\ _h$
1	0.1	0.1	10	1.0	91	0.0170368
2	0.1	0.01	100	1.0	7	0.0172243
3	0.01	0.01	100	1.0	62	0.00016576

The values of the norm of the total error (21) of algorithm for some variants are given in Table 2.

Example 3. Let

$$a = 0.3, \quad b = 0.2, \quad c = 1.0, \quad d = 0.5, \quad T = 1.0,$$

and

$$\begin{aligned} \alpha(x, t) &= \left(\frac{2}{5} + \frac{2}{15} \sin^2 \pi t - \pi^2 x(1 - x) \right) \sin \pi t \\ &\quad + \frac{1}{2}(1 + \exp(xt)(t(1 - x) - 1)), \\ \beta(x, t) &= \exp(xt) \left((x^2 - t^2 + \frac{1}{2})(1 - x) + 2t \right) \\ &\quad - \frac{1}{2}(1 - x + (1 - 2x) \sin \pi t), \end{aligned}$$

with the initial functions $w^{(1)}(x) = \pi x(1 - x)$, $w^{(2)}(x) = 0$, $\psi^{(1)}(x) = -x(x - 1)$, $\psi^{(2)}(x) = 0$.

An exact solution and functions (5) have the form $w = x(1 - x) \sin \pi t$, $\psi = (1 - x)(\exp(xt) - 1)$ and $u = \pi x(1 - x) \cos \pi t$, $v = (1 - 2x) \sin \pi t$, $f = x(1 - x) \exp(xt)$, $\varphi = 1 - \exp(xt)(1 + (x - 1)t)$, $\psi = (x - 1)(1 - \exp(xt))$.

The values of the norm of the total error (21) of algorithm for some variants are given in Table 3.

Table 3: Total error of the algorithm

No.	h	τ	n	t_n	m	$\ \Delta z_h^{n,m}\ _h$
1	0.1	0.1	5	0.5	10	0.0369772
2	0.05	0.05	10	0.5	20	0.0135773
3	0.025	0.025	40	1.0	20	0.0096951

Conclusion

In the paper, the dynamic behavior of the beam is described by using a nonlinear system of Timoshenko differential equations. For this model, the sought functions are the transverse deflection function and the function of the normal slope change. The goal was to construct an efficient numerical algorithm for an initial boundary value problem. The proposed algorithm was checked by means of the test examples and showed a satisfactory result.

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