THE SYSTEM OF TURBULENT FLOW EQUATIONS

K.F. Agamaliyeva (Rustamova)

Sumgait State University
AZ 5008-Sumgait, Azerbaijan
kama_faiq@mail.ru

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Abstract

It is necessary to note that phenomenological turbulence theories develop in the direction of obtaining such closing ratios that as parameters could possibly contain the constants being the same for the widest class of flows. This turbulence theory is quite widely used in practice as a basis for calculating characteristics of turbulent flows considering the fact that the closed systems of transfer equations that are managed to get within these theories are relatively simple and at the same time give good coincidence of calculated and experimental results.

Keywords and phrases: Turbulence, viscosity, diffusion, coefficient, dispersion.

Application of the obtained system of transfer equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = \bar{q} \text{ or } \frac{\partial \bar{u}_j}{\partial x_j} = \bar{q}, \tag{1}$$

$$\rho\left(\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_j}{\partial x_j}\right) = \rho \bar{F}_i + \frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{u}_j}{\partial x_j} - \rho \overline{u_i' u_j'}\right) + \left(\bar{u}_{*i} - \bar{u}_i\right)\bar{q}, \tag{2}$$

$$\rho c\left(\frac{\partial \bar{T}}{\partial t} + \bar{u}_j \frac{\partial \bar{T}}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial \bar{T}}{\partial x_j} - \rho c \overline{u_i' T'}\right) + \left(\bar{T}_* - \bar{T}\right)\bar{q} + \bar{D}_*, \tag{3}$$

for averaged quantities when studying turbulent flows allows to avoid to use information on irregular behavior of physical variables and at the same time from the mathematical point of view to simplify the problem of research. However, the system of transfer equations (1)-(3) is incomplete since its equations at turbulent flows contain a number of additional terms (turbulent stresses and turbulent heat flows) whose explicit forms are not known. Thus, there arises a nontrivial problem of closure of the system of transfer equations for averaged quantities that in turbulence theory is central. For that it is necessary to set up additional dependences or accept some kind of hypotheses on the relation between the seeming turbulent quantities (i.e. turbulent flows and heat flows) and the averaged flow parameters [8].

It is clear that mixing of macroscopic mass of medium inherent to turbulent flows informs these flows the ability to intensive diffusion transfer of substances contained in the medium. Wherefore, distributions of such "substances" as a pulse, heat, substances dissolved or weighted in medium are more smoothed in turbulent flow than in laminar one.

Detailed study of turbulent transfer processes shows that these processes are much more complex than molecular diffusion process. Nevertheless, to describe approximately these processes in the same way as was accepted in phenomenological theory of molecular diffusion, more exactly, to assume that the density of substance flow (i.e. the substance flow rate through unit area per unit time) is proportional to the substance gradient.

Boussinesq first suggested determining turbulent tangential stress (i.e. taken with the opposite sign of the pulse flow density) in plane-parallel flow along the axis x by the formula [3-4]:

$$-\rho\overline{u'_x u'_y} = \mu_T \frac{\partial \bar{u}_x}{\partial y}, \tag{4}$$

where μ_T is dynamic coefficient of turbulent viscosity (turbulent exchange). Boussinesq considered that the turbulent exchange coefficientμ_T, unlike the molecular viscosity coefficient μ_T is not a physical constant of the medium, it represents some space coordinate function, the function in a complex maner related to circumstances of the given turbulent flow.

Passing to the description of the spatial (three-dimensional) averaged motion, it is natural to generalize the Boussinesq turbulence hypothesis (4) similar to one as in the laminar flow equations the Newton hypothesis on viscous friction was generalized, i.e. to assume that the tensor of turbulent stresses is a homogeneous linear function of the tensor of averaged strain rates [3-4]:

$$-\rho\overline{u'_i u'_j} = \mu_T \bar{D}_{ij}, \tag{5}$$

where $\bar{D}_{ij}$ are components of the tensor averaged strain rates:

$$\bar{D}_{ij} = \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}. \tag{6}$$

It should be noted that formula (5) for determining $\rho\overline{u'_i u'_j}$ is efficient for flows in boundary layers and in smoothly changing flows with small Reynolds and Froude numbers, the turbulent energy release rate approximately equals its dissipation rate (3). In more complex and developed turbulent flows (i.e. for rather large Reynolds and Froude number), where the convective turbulence transfer is essential, in order to determine $\rho\overline{u'_i u'_j}$

academician A.N.Kolmogorov suggested the following generalized hypothesis

$$-\rho\overline{u_i'u_j'} = \mu_T \bar{D}_{ij} - \delta_{ij}\frac{2}{3}\rho k, \tag{7}$$

where δ_{ij} is Kronecker's symbol, k is kinetic energy of turbulence per unit mass:

$$k = 0,5\left(\overline{u_x'u_x'} + \overline{u_y'u_y'} + \overline{u_z'u_z'}\right) = 0,5\left(\overline{u_i'u_i'}\right). \tag{8}$$

Substituting formula (7) in dynamics equation (2), we get the system of phenomenological equations of the averaged turbulent flow of incompressible medium with external mass exchange (with source or drain of mass)

$$\begin{gathered}\rho\left(\frac{\partial\bar{u}_i}{\partial t} + \partial\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}\right) = \rho\bar{F}_i + \frac{\partial}{\partial x_i}\left(\bar{P} + \frac{2}{3}\rho k\right)\\ +\rho\left[\mu\tfrac{\partial}{\partial x_j}\left(D_{ij}\right) + \tfrac{\partial}{\partial x_j}\left(\mu_T\bar{D}_{ij}\right)\right] + \rho\left(\bar{u}_{*i} - \bar{u}_i\right)\bar{q}.\end{gathered} \tag{9}$$

The continuity equation of the averaged turbulent flow of incompressible medium with source (or drain) of mass keeps the form

$$\frac{\partial\bar{u}_i}{\partial x_i} = \bar{q}. \tag{10}$$

The system of equations (9) and (10) assuming in them $\left(\bar{u}_{*i} - \bar{u}_i\right)\bar{q} = 0$ and $\bar{q} = 0$ (i.e. in the absence of reactive force arising from attachment or detachment of the mass), matches exactly acad. A.N. Kolmogorov's dynamics equation for a turbulent flow of incompressible medium with constant mass [3]:

$$\begin{gathered}\rho\left(\frac{\partial\bar{u}_i}{\partial t} + \partial\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}\right) = \rho\bar{F}_i + \frac{\partial}{\partial x_i}\left(\bar{P} + \frac{2}{3}\rho k\right)\\ +\rho\left[\mu\frac{\partial}{\partial x_j}\left(D_{ij}\right) + \frac{\partial}{\partial x_j}\left(\mu_T\bar{D}_{ij}\right)\right],\end{gathered} \tag{11}$$

$$\frac{\partial\bar{u}_i}{\partial x_i} = 0. \tag{12}$$

To determine the kinetic energy of turbulence we use (according to $(k-\varepsilon)$ turbulence model) an equation in the form [7]

$$\rho\left(\frac{\partial k}{\partial t} + \bar{u}_j\frac{\partial k}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\mu_k}{P_{r_k}}\frac{\partial k}{\partial x_j}\right) + \mu_T\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right) - \rho\varepsilon, \tag{13}$$

where ε is turbulent energy dissipation rate

$$\begin{gathered}\rho\left(\frac{\partial\varepsilon}{\partial t} + \bar{u}_j\frac{\partial\varepsilon}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\mu_T}{P_{r_\varepsilon}}\frac{\partial\varepsilon}{\partial x_j}\right)\\ +\frac{c_1\mu_T\varepsilon}{k}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)\frac{\partial\bar{u}_i}{\partial x_j} - \frac{\rho c_2\varepsilon^2}{k}.\end{gathered} \tag{14}$$

Equations (13) and (14) are valid for $\mu_T >> \mu$.

Turbulent viscosity μ_T is expressed [1, 7] by local values of kinetic energy of turbulence k and turbulent energy dissipation rate ε in the following way:

$$\mu_T = \frac{c_0 \rho k^2}{\varepsilon}. \tag{15}$$

This turbulent viscosity μ_T is used to joint turbulent stresses. The empiric coefficients $c_0, c_1, c_2, P_{r_k}, P_{r_\varepsilon}$ contained in the above mentioned additional equations (13) and (14) equal [1]

$$c_0 = 0,09\,; c_1 = 1,45\,; c_2 = 1,90\,; P_{r_k} = 1,0\,; P_{r_\varepsilon} = 1,3. \tag{16}$$

It should be noted that there are also another turbulence models with two equations (i.e. to determine k andε), the Nga-Spolding, Wilcocks-Tracy models are the most used ones. Rebzin compared the $(k-\varepsilon)$ turbulence models for an incompressible medium and concluded that they all work well enough [1].

Thus, the system of averaged equations of dynamics (9) and continuity (10) with regard to additional relations of $(k-\varepsilon)$ turbulence model (13)-(15) represent a complete system of equations of motion of highly turbulent flow of incompressible medium with external source (or drain) of the mass.

In a number of applied problems, the so-called algebraic models of turbulence based on the Boussinesq hypothesis may be more effective [1]. For engineering calculations of many flows this hypothesis corresponds to reality with sufficient accuracy [7]. In slowly changing flows with small Froude numbers when normal turbulent stresses (it turns out that it is much smaller than other terms and they are simply ignored) may not be taken into account. In these conditions the difference between the Boussinesq hypothesis and Kolmogorov-Prandtle generalized theory disappears. Then the Kolmogorov-Prandtle hypothesis may be replaced by the simpler Boussinesq hypothesis [7].

It is known that in Boussinesq's phenomenological theory of turbulence it is assumed that the turbulent stresses tensor (by analogy of the Newton law on viscous friction in laminar flow) is a homogeneous function of averaged rates of deformation

$$\overline{u_i' u_j'} = \nu_T \bar{D}_{ij}, \tag{17}$$

where ν_T is a kinematic coefficient of turbulent viscosity.

Turbulent heat flow $\rho c \overline{u_j' T'}$ is connected with turbulent viscosity and averaged flow parameters. It is determined (by means of algebraic model in the form of Reynolds analogy based on similarity between heat transfer and pulse) by the formula

$$-\rho c\overline{u_j'T'} = \lambda_T \frac{\partial \bar{T}}{\partial x_j}, \tag{18}$$

where λ_T is a turbulent heat conductivity coefficient.

In turbulent flow the additional heat transfer $\rho c\overline{u_j'T'}$ is stipulated by turbulent flow. Experimental studies [1] confirm that the ratio of turbulent heat conductivity a_T to turbulent viscosity ν_T called Prandtle's turbulent number, $P_{r_T} = \frac{\mu_T c}{\lambda_T} = \frac{\nu_T}{a_T}$ is a function with "good" behavior. Usually, it is considered that $P_{r_T} = 0,9$. For the wall-mounted flows P_{r_T} changes from $0,5-0,7$ in the outer part of the boundary layer to $1,5$ near the wall. The turbulent heat flow $\rho c\overline{u_j'T'}$ is connected with turbulent viscosity μ_T and aversged flow parameters by means of the Prandtle turbulent number (P_{r_T}) in the following way:

$$-\rho c\overline{u_j'T'} = \frac{c\mu_T}{P_{r_T}} \frac{\partial \bar{T}}{\partial x_j} \tag{19}$$

or

$$-\overline{u_j'T'} = \frac{\nu_T}{P_{r_T}} \frac{\partial \bar{T}}{\partial x_j}. \tag{20}$$

From comparison (18) and (19) we have:

$$\lambda_T = \frac{c\mu_T}{P_{r_T}} = \frac{\nu_T}{P_{r_T}} \rho c. \tag{21}$$

Allowing for above expressions for the turbulent stress $-\rho c\overline{u_i'u_j'}$ and turbulent heat flow $-\rho c\overline{u_j'T'}$ we get a system of flow motion equation of a medium with heat mass exchange

$$\frac{\partial \bar{u}_i}{\partial x_i} = \frac{\partial \bar{u}_j}{\partial x_j} = \bar{q}, \tag{22}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \bar{F}_i - \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left[\left(1 + \nu_T/\nu\right)\bar{D}_{ij}\right] + \left(\bar{u}_{*i} - \bar{u}_i\right)\bar{q}, \tag{23}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_j \frac{\partial \bar{T}}{\partial x_j} = a \frac{\partial}{\partial x_j}\left[\left(1 + a_T/a\right)\frac{\partial \bar{T}}{\partial x_j}\right] + \left(\bar{T}_* - \bar{T}\right)\bar{q}. \tag{24}$$

Here a and a_T are the molecular and turbulent heat conductivity coefficients:

$$a = \lambda/\rho c = \nu/P_r, \tag{25}$$

$$a_T = \lambda_T/\rho c = \nu_T/P_{r_T}. \tag{26}$$

System (22)-(26) is not complete, i.e. ν_T and a_T are unknown. In mathematical modeling of turbulent flows the most complex problem is to determine turbulent transfer coefficients (to complete the system of turbulent flow motion equations), in particular, the turbulent viscosity coefficient ν_T contained in the dynamics equation (23), and turbulent heat conductivity coefficient a_T contained in the energy equation (24).

It was established [7] that a great majority of algebraic models of turbulence work well when the Prandtle turbulent number P_{r_T} is close to a unit, i.e. it is accepted $P_{r_T} = 1$ [1]. Therefore, provided $P_{r_T} = 1$ (by analogy with laminar flow where $P_r = 1$ is accepted), we can obtain $a = \nu$ and $a_T = \nu_T$. Then we can rewrite the system (22)-(24) in the following form [9]

$$\frac{\partial \bar{u}_i}{\partial x_i} = \frac{\partial \bar{u}_j}{\partial x_j} = \bar{q}, \tag{27}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \bar{F}_i - \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left[\left(1 + \nu_T/\nu\right)\bar{D}_{ij}\right] + \left(\bar{u}_{*i} - \bar{u}_i\right)\bar{q}, \tag{28}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_j \frac{\partial \bar{T}}{\partial x_j} = \nu \frac{\partial}{\partial x_j}\left[\left(1 + \nu_T/\nu\right)\frac{\partial \bar{T}}{\partial x_j}\right] + \left(\bar{T}_* - \bar{T}\right)\bar{q}. \tag{29}$$

To determine the turbulent viscosity coefficient we can use algebraic model of turbulence based on the Boussinesq hypothesis. One of the most successive models of this type was suggested by L.Prandtle

$$\nu_T = l^2 \frac{d\bar{u}}{dy}, \tag{30}$$

where l is the displacement (mixing) path length; $\bar{u}$ is an averaged rate component in the direction of the main flow; $\frac{d\bar{u}}{dy}$ is an averaged rate gradient; y is a transverse coordinate (distance from the wall).

Calculation of l contained in (30) depends on the type of the flow under consideration: a boundary layer, jet, trace, etc. For wall mounted flows (internal or external) good results are given by [2,7]

$$l = \chi y \left(1 - e^{-y/A}\right), \tag{31}$$

where$\chi = 0,41\,;\ A = 26$. The expression in the brackets $[1 - \exp\left(-y/A\right)]$ is Van Drist's damping function used to throw a bridge between completely developed boundary layer, where $l = \chi y$ and viscous sublayer, where $l \to 0$ [1].

Thus, allowing for (30) and (31), the system (27)-(29) is complete. It should be noted that algebraic models of turbulence have well established for relatively simple flows. And the structure of turbulence remains almost unchanged to the Max numbers$Ma < 5$. Conseqrently, change of physical features of the medium (density and other feature) need not to be taken into account in the equations of motoin used together with turbulence model. When solving some applied problems (for example, turbulent jet spreaded in unlimited space, or turbulence in atmosphere), behaving similar to Boussinesq, Townsend, Ibadzade, Makkaveev and others, turbulent viscosity coefficient may be replaced by its averaged value in flow's cross section (i.e. to accept $\nu_T = const$) and the system of equations of motion of turbulent flow of incompressible medium with heat mass exchange is simplified and takes the form:

$$\frac{\partial \bar{u}_i}{\partial x_i} = \bar{q}, \tag{32}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \bar{F}_i - \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + (\nu + \nu_T)\frac{\partial^2 \bar{\varepsilon}_{ij}}{\partial x_j^2} + (\bar{u}_{*i} - \bar{u}_i)\,\bar{q}, \tag{33}$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}_j \frac{\partial \bar{T}}{\partial x_j} = (\nu + \nu_T)\frac{\partial^2 \partial \bar{T}}{\partial x_j^2} + \left(\bar{T}_* - \bar{T}\right)\bar{q}. \tag{34}$$

In this form the turbulent motion equation coincides with the equation of incompressible medium with heat mass exchange. The difference is only in the viscosity value. Consequently, this system with regard to corresponding boundary conditions is a mathematical model of turbulent flow of incompressible viscous medium with heat mass exchange. Their analysis shows that in the absence of external sources (or drains) of mass $\bar{q} = 0$, the amount of motion $(\bar{u}_{*i} - \bar{u}_i)\,\bar{q} = 0$ and heat energy $\left(\bar{T}_* - \bar{T}\right)\bar{q} = 0$, and as a special case from them one can obtain the known equations of matematical model of averaged turbulent flow of incompressible viscous medium of constant mass [1, 6, 7].

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