EXPONENTIALLY CONVERGENT METHOD FOR ABSTRACT CAUCHY PROBLEM WITH NONLINEAR NONLOCAL CONDITION

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Abstract

Problem for the first order differential equation with an unbounded operator coefficient in Banach space and nonlinear nonlocal condition is considered. A numerical method is proposed and justified for the solution of this problem under assumption that the mentioned operator coefficient A is strongly positive and some existence and uniqueness conditions are fulfilled. The method is based on the reduction of the given problem to an abstract Hammerstein equation. The later one is discretized by collocation and then solved via the fixed–point iteration method. Each iteration of the method involves Sinc-based numerical evaluation of the operator exponential represented by a Dunford-Cauchy integral along hyperbola enveloping the spectrum of A. The integral part of nonlocal condition is approximated using the Clenshaw-Curtis quadrature formula.

Key words and phrases: Differential equation with an unbounded operator coefficient in Banach space, Nonlocal problem, Operator exponential.

AMS subject classification: 65J10, 65M70, 35K90, 35L90.

1 Introduction

The paper presents a numerical method for the nonlocal initial value problem

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) = 0, \quad t \in (-1,1), \\ &u(-1) - g\left(u(\cdot)\right) = u_0, \end{aligned} \tag{1}$$

where A is a linear sectorial operator with the dense domain $D(A)$ in the Banach space X and $g : C\left([-1,1]; X\right) \to X$ is a given operator function. Recall that the sectorialness of A means that its spectrum lies inside a sectorial domain $\Sigma \in \mathbb{C}$,

$$\Sigma\left(\rho,\varphi\right) = \left\{ z = \rho + r\mathrm{e}^{i\theta} : \ r \in [0,\infty),\, \rho \in \mathbb{R}_+,\, |\theta| < \varphi \right\}, \tag{2}$$

and the resolvent $R_A(z) = (zI - A)^{-1}$ of A satisfies the inequality

$$\|R_A(z)\| \le \frac{M}{1 + |z|}, \tag{3}$$

on the boundary of Σ and outside it. A unique pair of parameters (ρ_0, φ_0) such that $\Sigma(\rho_0, \varphi_0) \subseteq \Sigma(\rho, \varphi)$, for any other admissible pairs (ρ, φ), is called spectral parameters of A.

The aforementioned nonlocal problem (1) is well studied from the theoretical point of view. Initial investigations into the existence of solution to (1) is given in the works of Byszewski [5, 7] for the case when g is a function of the finite number of $u(t_i)$, $i = \overline{1, n}$. Further developments and generalizations can be found in [6, 23, 1, 29, 20] (see also the review in [22]). In this work we will use the existence conditions obtained in [10].

Available works on numerical methods for the nonlocal problems of type (1) usually deals with the particular realizations of operator A and nonlocal condition specific to some application area. Pioneering work of Bitsadze and Samarskii [4], inspired by the plasma research, deals with two-point nonlocal problem for elliptic partial differential operators. Similar two and three point nonlocal conditions appear in the works of Gordeziani [14, 15, 16] in applications of two dynamics of the thin-walled structures. In [26] Vabishchevich investigated the application of two-point nonlocal problems to the inverse problems of heat conduction (see also [3, 2]). Galerkin methods for multipoint nonlocal problems related to the diffusion processes were developed in the works of Canon with co-authors [8, 18].

Majority of the existing numerical methods are based on the finite difference or finite element approximations to the given differential equation. More novel numerical approaches presented in [12, 27, 28] are based on the direct approximation of the solution operator employing the Dunford-Cauchy formula and quadrature specifically adjusted to the spectral parameters of A. The later technique permits one to obtain natively parallelizable numerical methods with the accuracy automatically adjustable to the smoothness of initial data (methods without accuracy saturation). These two numerical properties are essential for the modern problem driven scientific simulations using the state-of-the-art multi-core computational architectures. Hence, we set the development of such a method for (1) as the main goal of the present work.

Section 2 serves a preparatory purpose. In this section the original problem is transformed to a more general nonlinear representation. In section 3 we show how to discretize the obtained nonlinear Hammerstein equation. The discretized problem is then studied in section 4, where we establish conditions on the existence of its solution with the help of the Banach fixed point theorem. The accuracy estimate for the error of fixed point iteration method is obtained therein. Final error estimate of the method is derived in section 5. The remaining part of this section is devoted to a numerical example, which demonstrates the method's effectiveness. Concluding remarks and possible extension are given in section 6.

2 Alternative representation of the given problem

The Hille-Yosida theorem ensures the existence of strongly continuous semigroup e^{-At}, if A is a sectorial operator with the dense domain $D(A) \subseteq X$. Furthermore, it is well known from the classical semigroup theory [24] that for such A the general solution to the abstract differential equation from (1) can be formally written as

$$u(t) = T(A,t)u(-1) \equiv \mathrm{e}^{-A(t+1)}u(-1), \quad \forall t \in [-1,\,1],\; u(-1) \in X.$$

When $u(-1) \notin D(A)$, the action $T(A,t)u(-1)$ should be understood in the sense of the limit. Next we incorporate the nonlocal condition given in (1) by substituting the expression for $u(-1)$ therefrom. It results in the general nonlinear problem

$$u(t) = T(A,t)\left[u_0 + g\left(u(\cdot)\right)\right]. \tag{4}$$

Equation (4) is commonly known as the abstract Hammerstein equation [17, 30]. Unlike (1) this equation is valid for any $u_0 \in X$, and becomes equivalent to (1), when $u(t)$ is differentiable and $u_0 \in D(A)$. The function satisfying (4) is called a mild solution of (1), while the original solution of (1) is called strong. Note that, by derivation of (4), both equations from (1) are incorporated into one formula, ready-made for the use of fixed point iteration. For that reason, either Banach [5] or Schauder [23] fixed point theorem can be directly applied to (4) resulting in the existence conditions for the mild solution (see [22, 23] for the detailed reviews of applicable techniques).

At first, it seems that equation (4) is not so valuable from the computational point of view, since it contains operator function $T(A,t)$. Its numerical evaluation is a non-trivial computationally involving task even in the case when A is a matrix [21].

In reality stable and efficient approximation of $T(A,t)$ is possible for $t \in [-1,\,1]$, as long as A is a linear sectorial operator with $\varphi < \frac{\pi}{2}$ [11]. Next we will summarize how to build the exponentially convergent approximation of $T(A,t)v$ for $v \in D(A^\alpha)$, $\alpha > 0$. The action of operator exponential $T(A,t)$ on v admits the following integral representation [9, 24]

$$\mathrm{e}^{-A(t+1)}v = \frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{-z(t+1)}R_A(z)v\,dz, \tag{5}$$

here $R_A(z) = (zI - A)^{-1}$ is the resolvent of A, and $\Gamma \in \mathbb{C} \setminus \Sigma$ stands for the integration contour, positively oriented with respect to the spectrum Σ of A. According to [11] the integration contour for the Dunford-Cauchy integral in (5) needs to be chosen as

$$\Gamma = \{z(s) = a_I \cosh(s) - ib_I \sinh(s) : \ s \in (-\infty, \infty)\}.$$

The unknown values of contour parameters a_I, b_I are to be determined in such a way that the integrand can be analytically extended into the strip $D_d \in \mathbb{C}$

$$D_d = \left\{ \xi = x + iy \,\middle|\, x \in \mathbb{R}, |y| < \frac{d}{2} \right\},$$

and remains bounded there. For a fixed A the strip width parameter $0 < d < \frac{\pi}{2}$ is responsible for the accuracy of the sinc-quadrature used below to approximate (5). The mentioned Sinc-quadrature provides exponentially convergent approximation of $T(A,t)$, $t > -1$ and attains its fastest convergence rate, when

$$d = \frac{\pi}{2} - \varphi.$$

For such d and (ρ_0, φ_0) the values of a_I, b_I are specified by the formulas

$$a_I = \rho_0 \frac{\cos\left(\frac{d}{2} + \varphi_0\right)}{\cos \varphi_0}, \quad b_I = \rho_0 \frac{\cos\left(\frac{d}{2} + \varphi_0\right)}{\cos \varphi_0}. \tag{6}$$

In addition to the performed parametrization of Γ we make the specific resolvent correction to (5) by putting

$$R_{A,1}(z) = R_A(z) - \frac{1}{z} I$$

in place of $R_A(z)$. Such correction compensates the poor decay of $T(A,-1)$ at infinity, and allows to guaranty the exponential convergence rate of the mentioned Sinc-quadrature [11]. After we conduct the described manipulations on (4) it will take the form

$$\begin{aligned} u(t) = &\int_{-\infty}^{\infty} e^{-z(\xi)(t+1)} z'(\xi) R_{A,1}(z(\xi)) \left[u_0 + g(u(\cdot))\right] \xi, \\ z'(\zeta) = &a_I \sinh \zeta - i b_I \cosh \zeta. \end{aligned} \tag{7}$$

Remark 2.1. *The reader might have noted that the performed correction can only be justified if the spectral shift ρ_0 is positive. Indeed for the negative ρ_0 both the correction and the definition of a_I, b_I have to be modified. On the other hand, one might always make spectral shift to be greter than zero by a simple transformation*

$$v(t) = e^{\rho_1 t} u(t), \quad \rho_1 > 0.$$

3 Discretization

The next step toward the fully discretized analogue of (4), relies upon the approximation of the integral in (7). In this section we utilize the quadrature based approximation developed in authors' earlier works (see [11] and the references therein). The operator exponential $\mathrm{e}^{-A(t+1)}v$ is approximated by $T_N(A,t)v$,

$$T_N(A,t)v = \frac{h}{2\pi i}\sum_{p=-N}^{N} \mathrm{e}^{-z(ph)(t+1)} z'(ph) R_{A,1}(z(ph))v. \tag{8}$$

Expression on the right of (8) is obtained from the parametrized version of Dunford-Cauchy integral (7), with help of the Sinc-quadrature(trapezoid) formula. Before we state the result concerning the accuracy of the above approximation, let us highlight that all the summands in (8) are mutually independent. The evaluation of every summand involves the calculation of the time-dependent scalar part and the evaluation of resolvent part, free of t. Computationally it means that the resolvent evaluations $R_{A,1}(z(ph))v$, $p = \overline{-N, N}$ can be performed in parallel and the results are stored. Once that is done all the evaluation of $T_N(A,t)v$ – for as many $t \in [-1,1]$ as needed – can be achieved at the fraction of cost, spent for the resolvent evaluations. This is especially true if the evaluation of $R_{A,1}(z(ph))v$ predominates the calculation of the scalar part in terms of computational complexity. Some additional savings of computational resources are possible if the operator A is real-valued [12]. The accuracy of the proposed approximation is characterized by the following theorem [11, p. 34].

Theorem 1. *Assume that A is a linear sectorial operator with the densely defined domain and $v \in D(A^\alpha)$. Let*

$$h = \sqrt{\frac{\pi d}{\alpha(N+1)}},$$

then the error $\eta_N(t)v \equiv \|T(A,t)v - T_N(A,t)v\|$ satisfies the estimate

$$\|\eta_N(t)v\| = \left\|\mathrm{e}^{-A(t+1)}v - T_N(A,t)v\right\| \le \frac{c\,\mathrm{e}^{-\sqrt{\pi d\alpha(N+1)}}}{\alpha}\|A^\alpha v\|, \tag{9}$$

with some positive constant c independent of A, v, α and t.

Estimate (9) demonstrates that the approximant $T_N(A,t)v$ meets all the requirements formulated in section 2. Bearing that in mind we proceed to the collocation of (4).More precisely we will apply the polynomial collocation method to the modified version thereof

$$u(t) = T_N(A,t)\left[u_0 + g\left(u(\cdot)\right)\right]. \tag{10}$$

Let us introduce the Chebyshev-Gauss-Lobatto (CGL) nodes $t_j = -\cos\left(\frac{\pi j}{n}\right)$, $j = \overline{0,n}$. For a given vector $\vec{y} = (y_0, \ldots, y_n)$ we define the modified Hermite-Fejér polynomial [25]

$$K_{2n-1}(t, \vec{y}) = \sum_{i=0}^{n} B_{i,2n-1}(t) y_i,$$

of the degree $2n-1$. For each $i = 0, 1, \ldots n$, $B_{i,2n-1}$ is the unique polynomial such that $B_{i,2n-1}(t_j) = \delta_{i,j}$, for $j = \overline{0,n}$, and $B'_{i,2n-1}(t_j) = 0$, for $j = \overline{1, n-1}$

$$B_{0,2n-1}(t) = \frac{1+t}{2} P_{n-1}^2(t), \quad B_{0,2n-1}(t) = \frac{1-t}{2} P_{n-1}^2(t)$$
$$B_{i,2n-1}(t) = \frac{\left(1-t^2\right)\left(1+tt_i-2t_i^2\right)}{(n-1)^2 (t-t_i)^2 \left(1-t_i^2\right)} P_{n-1}^2(t), \quad i = \overline{1, n-1},$$

where $P_n(t)$ is Chebyshev polynomial of the first kind. Now, we put the polynomial $K_{2n-1}(t, \vec{y})$ in place of $u(t)$ in (10) and collocate the received equation at the sequence of interpolation points. It leads us to the system of nonlinear equations

$$y_i = T_N(A, t_i) u_0 + T_N(A, t_i) g\left(K_{2n-1}(\cdot, \vec{y})\right), \quad i = \overline{0,n}, \tag{11}$$

with respect to the unknowns y_i. Similarly to (4) this system, can be directly used to find the approximation to $u(t)$ on the chosen grid, since $y_i = u(t_i)$ is clearly the solution. In the following section we are going to theoretically justify the iterative solution method based upon (11).

4 Solution of discretized problem

In order to prove the existence of solution to (11) we recast this system in a vector-matrix form

$$\vec{y} = \vec{g}\left(\vec{y}\right) + \vec{p}, \tag{12}$$

where

$$\begin{aligned} \vec{g}(\vec{y}) = & \left(T_N(A, t_0) g(K_{2n-1}(\cdot, \vec{y})), \ldots, T_N(A, t_n) g\left(K_{2n-1}(\cdot, \vec{y})\right)\right)^T, \\ \vec{p} = & \left(T_N(A, t_0) u_0, \ldots, T_N(A, t_n) u_0\right)^T. \end{aligned}$$

For the existence of the solution it is sufficient to show that a recurrence sequence

$$\vec{y}^{(k)} = \vec{g}\left(\vec{y}^{(k-1)}\right) + \vec{p}, \quad \vec{y}^{(0)} = \vec{p}, \tag{13}$$

is convergent in the vector space $X^n = X \times X \times \ldots X$. For any $\vec{x} \in X^n$ let us introduce a norm

$$|||\vec{x}||| = \max_{0 \le j \le n} \|x_j\|.$$

Regarding the function g we require to satisfy the following Lipschitz-like condition for any $u, v \in C([-1,1]; X)$

$$\|A^{\alpha}(g(u) - g(v))\| \le L \max_{t \in [-1,1]} \|u(t) - v(t)\|, \tag{14}$$

with some positive constants α and $L < \infty$. Apart from (14) we will use the estimate

$$\left\|T_N(A, t_i) A^{-\alpha}\right\| \le \frac{c}{\alpha}, \tag{15}$$

which is a mere consequence of (3.278) from [11].

Theorem 2. *Assume that $A : X \to X$ is a linear operator satisfying the condition of theorem 1 and $g : C([-1,1]; X) \to X$ is an operator function satisfying* (14)*. If there exist such $L, c, \alpha > 0$ from* (14),(15)*, that*

$$q \equiv \frac{3Lc}{\alpha} < 1, \tag{16}$$

then equation (4) *has a unique solution $\vec{y}^{(\infty)} = \lim_{k \to \infty} \vec{y}^{(k)}$. Moreover*

$$\left|\left|\left|\vec{y}^{(\infty)}\right|\right|\right| \le |||\vec{p}||| + |||\vec{g}(\vec{p})||| \frac{1}{1-q}, \tag{17}$$

and an error of the k-th iterative approximation admits the estimate

$$\left|\left|\left|\vec{y}^{(\infty)} - \vec{y}^{(k)}\right|\right|\right| \le |||\vec{g}(\vec{p})||| \frac{q^{k+1}}{1-q}. \tag{18}$$

Proof. To show that $y^{(\infty)}$ is a unique solution of (12) we apply the Banach fixed point theorem. The space X^n equipped with the metric $d(x, y) = |||\vec{x} - \vec{y}|||$ forms a complete Banach space. The mapping $\mathcal{F}$ defined by (13) transforms the space X^n into itself. To demonstrate existence of the fixed point it remains to show that this mapping is contractive.

$$\begin{aligned}
&\left|\left|\left|\mathcal{F}\vec{\S} - \mathcal{F}\vec{\dagger}\right|\right|\right| = |||\vec{g}(\vec{x}) - \vec{g}(\vec{y})||| \\
\le &\max_{0 \le j \le n} \left\|T_N(A, t_j) A^{-\alpha}\right\| \left\|A^{\alpha}\left(g\left(K_{2n-1}(\cdot, \vec{x})\right) - g\left(K_{2n-1}(\cdot, \vec{y})\right)\right)\right\| \\
\le &\frac{c}{\alpha} \left\|A^{\alpha}\left(g\left(K_{2n-1}(\cdot, \vec{x})\right) - g\left(K_{2n-1}(\cdot, \vec{y})\right)\right)\right\|
\end{aligned}$$

We used (15), to get the last estimate. For any $\vec{x} \in X^n$ the polynomial $K_{2n-1}(t, \vec{x})$ is, obviously, a continuous function of t. Moreover, it was proved in [25], that

$$\max_{t\in[-1,1]} \sum_{i=0}^{n} |B_{i,2n-1}(t)| < 3, \quad n = 0, 1, 2, \dots$$

These two observations together permit us to write

$$\left|\left|\left|\mathcal{F}\vec{\S} - \mathcal{F}\vec{\dagger}\right|\right|\right| \le \frac{Lc}{\alpha} \max_{t\in[-1,1]} \sum_{i=0}^{n} |B_{i,2n-1}(t)| \, \|x_i - y_i\| \le \frac{3Lc}{\alpha} |||\vec{x} - \vec{y}|||. \tag{19}$$

The contraction of $\mathcal{F}$ is proved.

Since $\vec{y}^{(k)} = \sum_{l=1}^{k} (\vec{y}^{(l)} - \vec{y}^{(l-1)}) + \vec{p}$, we are actually interested in the difference of two consecutive elements of the sequence generated by (13). The estimate for $\left|\left|\left|\vec{y}^{(k)} - \vec{y}^{(k-1)}\right|\right|\right|$ is provided by (19), specifically

$$\left|\left|\left|\vec{y}^{(k)} - \vec{y}^{(k-1)}\right|\right|\right| = \left|\left|\left|\vec{g}\left(\vec{y}^{(k-1)}\right) - \vec{g}\left(\vec{y}^{(k-2)}\right)\right|\right|\right| \le q \left|\left|\left|\vec{y}^{(k-1)} - \vec{y}^{(k-2)}\right|\right|\right|. \tag{20}$$

So, in the end, it all goes down to

$$\left|\left|\left|\vec{y}^{(1)} - \vec{y}^{(0)}\right|\right|\right| = |||\vec{g}\,(\vec{p})|||$$

To justify (18) we apply (20) to every term in the series for $\vec{y}^{(k)}$.

$$\left|\left|\left|y^{(k)}\right|\right|\right| \le |||\vec{p}||| + \sum_{p=0}^{k} \left(\frac{3Lc}{\alpha}\right)^p |||\vec{g}\,(\vec{p})||| = |||\vec{p}||| + |||\vec{g}\,(\vec{p})||| \frac{1 - q^{k+1}}{1 - q}. \tag{21}$$

Inequality (17) can be derived analogously to (21).

5 Error analysis

The developed method emerges as a combination of three different numerical procedures: spectral approximation of the operator exponential $T_N(A, t)$, collocation–based discretization and iterative solution of the discretized problem. In this section we derive the compound error estimate by analysing the error contribution from the every listed numerical procedure.

Let $z_i = u(t_i) - y_i$, $i = \overline{0, n}$ is the pointwise difference of solutions to (4) and (12). Denote $\vec{u} = (u(t_0), \dots, u(t_n))^T$, $\vec{z} = (z_0, \dots, z_n)^T$. Then, evaluate the quantity z_i using (4) and (11)

$$\begin{aligned} z_i =& T(A, t_i)u_0 - T_N(A, t_i)u_0 + T(A, t_i)g\,(u(\cdot)) - T_N(A, t_i)g\,(K_{2n-1}(\cdot, \vec{y})) \\ =& T(A, t_i)u_0 - T_N(A, t_i)u_0 + T(A, t_i)g\,(u(\cdot)) - T(A, t_i)g\,(K_{2n-1}(\cdot, \vec{y})) \\ &+ T(A, t_i)g\,(K_{2n-1}(\cdot, \vec{y})) - T_N(A, t_i)g\,(K_{2n-1}(\cdot, \vec{y})) \\ =& \eta_N(t_i)u_0 + \eta_N(t_i)g\,(K_{2n-1}(\cdot, \vec{y})) + T(A, t_i)\left[g\,(u(\cdot)) - g\,(K_{2n-1}(\cdot, \vec{y}))\right] \end{aligned}$$

The terms $\eta_N(t_i)u_0$, $\eta_N(t_i)g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right)$ can be estimated by (9), provided that the conditions of theorem 1 are fulfilled. For convenience we denote

$$\mu_n = T(A,t_i)\left[g\left(u(\cdot)\right) - g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right)\right],$$

and then estimate the third term

$$\begin{aligned}
|||\mu_n||| &= \max_{0\le i\le n}\left\|T_N(A,t_i)\left[g\left(u(\cdot)\right) - g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right)\right]\right\| \\
&\le \max_{0\le i\le n}\left\|T_N(A,t_i)A^{-\alpha}\right\|\left\|A^{\alpha}\left[g\left(u(\cdot)\right) - g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right)\right]\right\| \\
&\le \frac{Lc}{\alpha}\max_{t\in[-1,1]}\left\|u(\cdot) - K_{2n-1}\left(\cdot,\vec{y}\right)\right\| \le \frac{Lc}{\alpha}\left(\Lambda_{2n-1}(u) + 3|||\vec{z}|||\right)
\end{aligned}$$

where $\Lambda_{2n-1}(u)$ is an error of the approximation of u by the modified Hermit-Fejér polynomial of degree $2n-1$. It is well know that, unlike Lagrange interpolation, the Hermite-Fejér interpolation is convergent for any $u \in C\left([-1,1];X\right)$ [25]. The resulting estimate of $|||z|||$ is given by the lemma below.

Lemma 3. *Assume that the conditions of theorem 2 is met and, in addition, that there exist $\alpha > 0$ such that $u_0, g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right) \in D(A^{\alpha})$, $\forall\vec{y} \in X^n$. Then the solution to discretized system (11) gives a pointwise approximation of the solution to Hammerstein equation (4). The approximation error satisfies the following estimate*

$$\begin{aligned}
&|||z||| \le \tfrac{1}{1-q} \\
&\times\left(\tfrac{c\,\mathrm{e}^{-\sqrt{\pi d\alpha(N+1)}}}{\alpha}\left(\|A^{\alpha}u_0\| + \left\|A^{\alpha}g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right)\right\|\right) + \tfrac{Lc}{\alpha}\Lambda_{2n-1}(u)\right).
\end{aligned} \tag{22}$$

Proof. Most of the proof was presented before the lemma. To prove the final error estimate we combine the estimate for $|||\mu_n|||$ with the results of theorem 1 for $\eta_N(t_i)u_0$, $\eta_N(t_i)g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right)$, $i=\overline{0,n}$.

The compound error estimate of the method is given by the following theorem.

Theorem 4. *Assume that the operator A and function g satisfy the conditions of theorem 2 and, in addition, that $u_0, g\left(K_{2n-1}\left(\cdot,\vec{y}\right)\right) \in D(A^{\alpha})$, $\forall\vec{y} \in X^n$. Then the k-th iterative approximation given by (13) constitutes a pointwise approximation of the solution to Hammerstein equation (4). The error of this approximation satisfies the inequality*

$$\left|\left|\left|\vec{u} - \vec{y}^{(k)}\right|\right|\right| \le |||\vec{z}||| + c_1 M_C\frac{q^{k+1}}{1-q}|||\vec{p}|||, \tag{23}$$

where $M_C = \sup\limits_{v\in C([-1,1];X)}\frac{\|g(v)\|}{\|v\|}$, and $c_1 > 0$ is independent of g.

Remark 5.1. *Inequality* (23) *along with the known estimates of* $\Lambda_{2n-1}(u)$ *[13] demonstrate that the order of the error of Hermite-Fejér interpolation is lower, than other contributions to the compound error estimate. To improve the overall convergence one might use other interpolation technique to collocate* (10)*, like spline interpolation of odd degree [19], for instance. The compound numerical method could adapt any other interpolation operator with the bounded norm, as long as its image is in* $C([-1,1];X)$*.*

In spite of the fact that $\Lambda_{2n-1}(u) \sim \ln n/n$ as $n \to \infty$ it does not present a real challenge from the computational point of view, since the computational complexity of the right-hand side evaluation from (13) grows very slowly when n increases. It happens because of the mentioned in section 3 computational properties of the operator exponential approximation.

Example 5. Here we experimentally consider the specification of problem (1), with A being a second order elliptical operator:

$$Au = -\frac{\partial^2 u}{\partial x^2}, \quad u(0) = u(1) = 0,$$

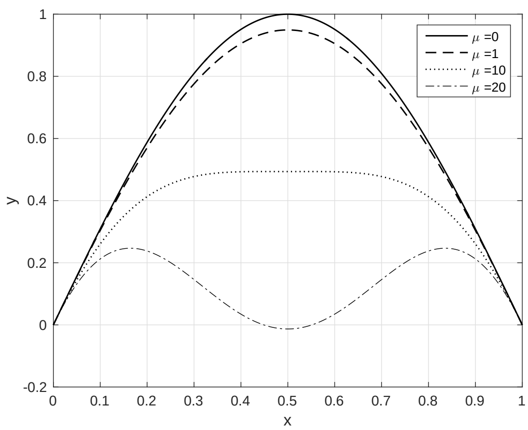

Figure 1. Graph of $u_0(x)$ for the different values of μ

and such, that the nonlocal condition is defined by

$$\begin{aligned} &u(-1) - \mu \int_{-1}^{1} u^2(s)ds = u_0, \\ &u_0(x) = \sin(\pi x) + \mu \frac{e^{-4\pi^2} - 1}{2\pi^2} \sin^2(\pi x). \end{aligned} \tag{24}$$

To benchmark the developed numerical method for the different values of q (16) we have set $g(u(\cdot)) = \mu \int_{-1}^{1} u^2(s)ds$. This expression contains one additional parameter μ, which enables us to change the size of q through

L. Exact solution u_{ex} to the considered problem can be written as

$$u_{ex}(t,x) = e^{-(t+1)\pi^2} \sin(\pi x). \tag{25}$$

Note, that $u_{ex}(t,x)$ does not depend on μ and is equal to $u_0(x)$, when $t = 0$, $\mu = 0$. One can observe from the graph of u_0, depicted in Fig. 1, that the solution of nonlocal problem is close to the solution of the corresponding classical Cauchy problem (with the initial condition $u(-1) = u_0$) the difference between these two solutions grows if μ gets bigger (compare the graphs of $u_0(x)$ for $\mu = 0$ and $\mu = 20$ from Fig. 1).

To measure an experimental accuracy of the method we define

$$Err = \mathrm{Err}\left(\mathrm{u_{ex}}, \tilde{\mathrm{y}}^{(\mathrm{k})}\right) \equiv \max_{0\le \mathrm{l} \le \mathrm{m}} \max_{0\le \mathrm{j} \le \mathrm{n}} \left\| \mathrm{u_{ex}}(\mathrm{t_i}, \mathrm{x_l}) - \mathrm{y_j^{(k)}}(\mathrm{x_l}) \right\|,$$

where $x_l = \frac{1}{2}\left(1 - \cos\left(\frac{\pi l}{m}\right)\right)$, $l = \overline{0, m}$ is the scaled version of CGL nodes. After this is done, one needs to define the stopping criteria for iterative the method (13). For that matter we shall use

$$\mathrm{Err}\left(\tilde{\mathrm{y}}^{(\mathrm{k})}, \tilde{\mathrm{y}}^{(\mathrm{k-1})}\right) < 10^{-18},$$

to factor out the influence of the iterative error.

Initially, we set $\mu = 1/4$. Such a choice of μ guarantees the validity of (16), for $\forall \rho_0 \in \mathbb{R}_+$, $\alpha = 1$. Moreover, function $g(\cdot)$ admits, as a function of scalar variable, analytic extension into the region $\mathcal{E}_\rho \in \mathbb{C}$. It means that all the suppositions of theorems 2 and 4 are met. Consequently, the iterative solution of (12) exists and can be approximated by $\vec{y}^{(k)}$. This solution converges to the initial problem's solution $u(t,x)$ $(n, N \to \infty)$. The experimental results, calculated by (13), (8), a presented in Table 1 for the different values of N, n. For each $N \in \{4, 8, 16, 32, 64, 128, 256\}$, we experimentally selected n sufficiently large for the error Err(n) to saturate (see Fig. 2a).

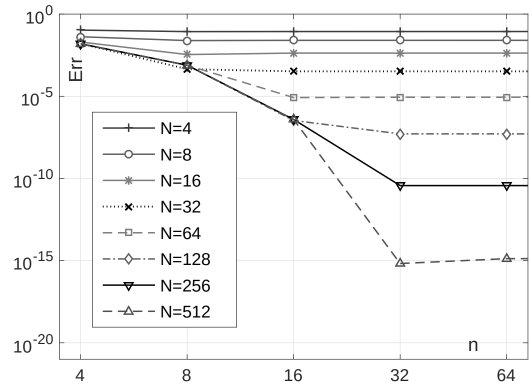

Figure 2a, $\mu = 0.25$

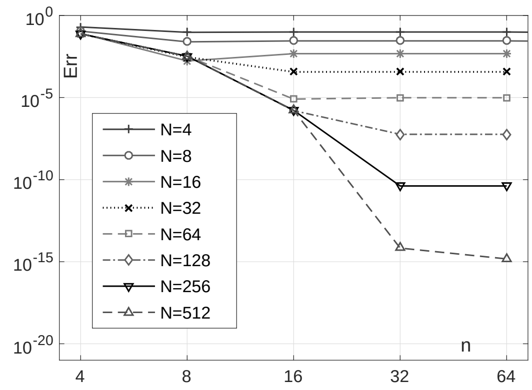

Figure 2b, $\mu = 1$

Graph of experimental error Err as a function of the number of collocation points n, drawn in the logarithmic scale for $N = 4, 8, 16, 32, 64, 128, 256, 512$.

Those two are shown in Table 1 alongside the value of Err and the number of iterations (denoted by K) needed to achieve the iteration's relative accuracy 10^{-5}, estimated a posteriori. As observed from Table 1, the experimental converge of the developed method is exponential with respect to n.

N	$\ln n$	Err	K
4	8	0.0859119243400000010	3
8	8	0.0244950525900000000	3
16	8	0.00345794666699999987	3
32	16	0.000328787487900000005	3
64	16	0.0000083384394889999922	3
128	32	0.00000005155130762999999962	4
256	32	$3.68083566999999982 \times 10^{-11}$	4
512	64	$1.32334447899999999 \times 10^{-15}$	4

Table 1: Result of experimental application of the developed method to the numerical solution of (1)

The preposition of theorem 2 is no longer true for $\mu = 1$. In spite of that the method is still convergent. Several graphs of Err as a function of n are depicted in Fig. 2b. They demonstrate a qualitatively similar exponentially decreasing behaviour of the experimental error.

6 Conclusions and future work

In this work we developed and justified the method for the numerical solution of abstract nonlocal Cauchy problem (1). If the operator function $g\left(u(\cdot)\right)$ is continuous and A is sectorial with the angle $\phi_0 < \frac{\pi}{2}$ is globally convergent. Already being general, problem (1) can be generalized even more by adding the nonlinear right-hand side. Existing theoretical results for such problems [1, 10] suggest that the extension of developed methodology to that class of problems is possible. This consists the topic for the future work.

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