# THE ROBIN PROBLEM FOR THE HELMHOLTZ EQUATION IN A THREE-DIMENSIONAL STARLIKE DOMAIN 

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Abstract

The internal and external Robin problems for the Helmholtz equation in bounded starlike domains are addressed. We show how to derive the relevant solution by using a suitable Fourier series-like method. Numerical results are specifically obtained considering three-dimensional domains whose boundary is defined by a generalization of the so-called "superformula" introduced by Gielis. By using the computer algebra code Mathematica ${ }^{\circledR}$, truncated series approximations of the solutions are determined. Our findings are in good agreement with the theoretical results on the Fourier series due to Carleson.

Key words and phrases: Robin problem, Helmholtz equation, Starlike domain, Fourier series.

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## 1 Introduction

Many applications of the mathematical physics and electromagnetics are related to the Laplacian differential operator. Among them it is worth mentioning those relevant to the wave equation, the Laplace and Poisson equations, the Helmholtz equation, as well as the Schrödinger equation. However, the most part of the boundary-value problems ( $B V P \mathrm{~s}$ ) relevant
to the Laplacian can be solved in an explicit form only in domains with very special shape or symmetries, namely intervals, cylinders or spheres, [1].

The solution in more general domains can be obtained by using the Riemann theorem on conformal mappings, and the relevant invariance of the Laplacian [2]. However, explicit conformal mappings are known only for particular domains and, of course, such a method can not be applied in the three-dimensional case, where approaches based on a suitable spatial discretization procedure are usually adopted.

Different techniques have been proposed for solving the general problem both from a theoretical and numerical point of view (e.g., representing the solution by using boundary layer techniques [3]; solving by iterative methods the corresponding boundary integral equation [4]; approximating the relevant Green function by the least squares method [5]; solving linear systems relevant to elliptic partial differential equations by relaxation methods [6]). Anyway, none of the contributions already available in the scientific literature deals with the here developed approach, which makes use of simple analytical tools tracing back to the original Fourier projection method [7].

We consider in this paper an extension of the classical theory regarding the Robin problem for the Helmholtz equation in a starlike domain, i.e. a domain $\mathcal{D}$ which is normal with respect to a suitable spherical coordinate system so that the relevant boundary $\partial \mathcal{D}$ may be regarded as an anisotropically stretched unit sphere [8]-[10]. An efficient technique useful to compute the coefficients of the Fourier-like expansion approximating the solution of the Helmholtz equation in such a domain is proposed. Regular functions are considered for the boundary data, but the presented theory can be easily extended by considering weakened hypotheses. Furthermore, a generalization of the so called "superformula" due to Gielis [11] is used to define the boundary of the domains considered in the present research.

Several numerical examples, addressed by means of the computer algebra code Mathematica ${ }^{\odot}$, have shown a point-wise convergence of the solution with possible oscillations occurring in cusped or quasi-cusped points of the domain boundary in agreement with the theoretical findings by Carleson [13].

## 2 The Laplacian in stretched spherical co-ordinates

Let us introduce in the three-dimensional space the usual spherical coordinate system

$$
\begin{equation*}
x=r \sin \vartheta \cos \varphi, \quad y=r \sin \vartheta \sin \varphi, \quad z=r \cos \vartheta, \tag{2.1}
\end{equation*}
$$

and assume that the boundary of the normal domain $\mathcal{D}$ is described by the polar equation

$$
\begin{equation*}
r=R(\vartheta, \varphi), \tag{2.2}
\end{equation*}
$$

where $R(\vartheta, \varphi)$ is a $C^{2}$ function for $(\vartheta, \varphi) \in[0, \pi] \times[0,2 \pi]$. Therefore, in the interior of $\mathcal{D}$ the following inequality is satisfied

$$
\begin{equation*}
r \leq R(\vartheta, \varphi) \tag{2.3}
\end{equation*}
$$

Furthermore, as it can be easily inferred, the consistency condition $\min _{(\vartheta, \varphi)}$ $R(\vartheta, \varphi)>0$ must hold. Let us then define the stretched radius $\rho$ in such a way that

$$
\begin{equation*}
r=\rho R(\vartheta, \varphi) . \tag{2.4}
\end{equation*}
$$

In this way, naturally induced in the $x, y, z$ space are the curvilinear (namely, stretched) co-ordinates $\rho, \vartheta, \varphi$ related to the cartesian ones by

$$
\begin{gather*}
x=\rho R(\vartheta, \varphi) \sin \vartheta \cos \varphi, \quad y=\rho R(\vartheta, \varphi) \sin \vartheta \sin \varphi,  \tag{2.5}\\
z=\rho R(\vartheta, \varphi) \cos \vartheta .
\end{gather*}
$$

As a consequence, $\mathcal{D}$ is obtained by assuming $0 \leq \vartheta \leq \pi, 0 \leq \varphi \leq 2 \pi$, $0 \leq \rho \leq 1$.

Remark 1. Note that in the stretched co-ordinate system the original domain $\mathcal{D}$ is transformed into the unit sphere. As a result, in such a system classical techniques, including the separation of variables, can be used for solving the transformed Helmholtz equation.

We consider a $C^{2}(\mathcal{D})$ function $v(x, y, z)=v(r \sin \vartheta \cos \varphi, r \sin \vartheta \sin \varphi$, $r \cos \vartheta)=u(r, \vartheta, \varphi)$ and the Laplace operator in spherical co-ordinates

$$
\begin{equation*}
\Delta u=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial u}{\partial r}\right)+\frac{1}{r^{2} \sin \vartheta} \frac{\partial}{\partial \vartheta}\left(\sin \vartheta \frac{\partial u}{\partial \vartheta}\right)+\frac{1}{r^{2} \sin ^{2} \vartheta} \frac{\partial^{2} u}{\partial \varphi^{2}} . \tag{2.6}
\end{equation*}
$$

We can readily represent such a differential operator in the stretched coordinate system $\rho, \vartheta, \varphi$. By setting

$$
\begin{equation*}
U(\rho, \vartheta, \varphi)=u(\rho R(\vartheta, \varphi), \vartheta, \varphi) \tag{2.7}
\end{equation*}
$$

and denoting for shortness $R(\vartheta, \varphi):=R$, we find after some mathematical manipulations

$$
\begin{gather*}
\Delta u=\frac{\partial^{2} u}{\partial r^{2}}+\frac{2}{r} \frac{\partial u}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \vartheta^{2}}+\frac{\cot \vartheta}{r^{2}} \frac{\partial u}{\partial \vartheta}+\frac{1}{r^{2} \sin ^{2} \vartheta} \frac{\partial^{2} u}{\partial \varphi^{2}} \\
=\frac{1}{R^{2}}\left(1+\frac{R_{\vartheta}^{2}}{R^{2}}+\frac{R_{\varphi}^{2}}{R^{2} \sin ^{2} \vartheta}\right) \frac{\partial^{2} U}{\partial \rho^{2}} \\
+\frac{1}{\rho R^{2}}\left[2\left(1+\frac{R_{\vartheta}^{2}}{R^{2}}+\frac{R_{\varphi}^{2}}{R^{2} \sin ^{2} \vartheta}\right)-\frac{1}{R}\left(R_{\vartheta} \cot \vartheta+R_{\vartheta \vartheta}+\frac{R_{\varphi \varphi}}{\sin ^{2} \vartheta}\right)\right] \frac{\partial U}{\partial \rho} \\
-2 \frac{R_{\vartheta}}{\rho R^{3}} \frac{\partial^{2} U}{\partial \rho \partial \vartheta}-2 \frac{R_{\varphi}}{\rho R^{3} \sin ^{2} \vartheta} \frac{\partial^{2} U}{\partial \rho \partial \varphi}+\frac{1}{\rho^{2} R^{2}} \frac{\partial^{2} U}{\partial \vartheta^{2}} \\
+\frac{\cot \vartheta}{\rho^{2} R^{2}} \frac{\partial U}{\partial \vartheta}+\frac{1}{\rho^{2} R^{2} \sin ^{2} \vartheta} \frac{\partial^{2} U}{\partial \varphi^{2}} . \tag{2.8}
\end{gather*}
$$

As it can be noticed, for $\rho=r, R(\vartheta, \varphi) \equiv 1$ the Laplacian in spherical co-ordinates is recovered.

## 3 The Robin problem for the Helmholtz equation

Let us consider the internal Robin problem for the Helmholtz equation in the starlike domain $\mathcal{D}$ having boundary described by the polar equation $r=R(\vartheta, \varphi)$,

$$
\begin{cases}\Delta v(x, y, z)+k^{2} v(x, y, z)=0, & (x, y, z) \in \dot{\mathcal{D}}  \tag{3.1}\\ \gamma v(x, y, z)+\lambda \frac{\partial v}{\partial \nu}(x, y, z)=f(x, y, z), & (x, y, z) \in \partial \mathcal{D}\end{cases}
$$

where $k>0$ denotes the wave-number, $\gamma \neq 0, \lambda$ are arbitrary constants, and $\hat{\nu}=\hat{\nu}(\vartheta, \varphi)$ is the outward-pointing normal to $\partial \mathcal{D}$. We can easily prove the following theorem.

Theorem 3.1 Let

$$
\begin{gather*}
\psi_{\vartheta}(\vartheta, \varphi)=\frac{R_{\vartheta}(\vartheta, \varphi)}{R(\vartheta, \varphi)}  \tag{3.2}\\
\psi_{\varphi}(\vartheta, \varphi)=\frac{R_{\varphi}(\vartheta, \varphi)}{R(\vartheta, \varphi) \sin \vartheta}  \tag{3.3}\\
\psi(\vartheta, \varphi)=\sqrt{\psi_{\vartheta}(\vartheta, \varphi)^{2}+\psi_{\varphi}(\vartheta, \varphi)^{2}} \tag{3.4}
\end{gather*}
$$

and

$$
\begin{gather*}
f(R(\vartheta, \varphi) \sin \vartheta \cos \varphi, R(\vartheta, \varphi) \sin \vartheta \sin \varphi, R(\vartheta, \varphi) \cos \vartheta)=F(\vartheta, \varphi) \\
\quad=\sum_{n=0}^{+\infty} \sum_{m=0}^{n} P_{n}^{m}(\cos \vartheta)\left(\alpha_{n, m} \cos m \varphi+\beta_{n, m} \sin m \varphi\right) \tag{3.5}
\end{gather*}
$$

where

$$
\begin{align*}
& \left\{\begin{array}{l}
\alpha_{n, m} \\
\beta_{n, m}
\end{array}\right\}=\epsilon_{m} \frac{2 n+1}{4 \pi} \frac{(n-m)!}{(n+m)!} \int_{0}^{2 \pi} \int_{0}^{\pi} F(\vartheta, \varphi) P_{n}^{m}(\cos \vartheta)  \tag{3.6}\\
& \times\left\{\begin{array}{c}
\cos m \varphi \\
\sin m \varphi
\end{array}\right\} \sin \vartheta d \vartheta d \varphi,
\end{align*}
$$

$\epsilon_{m}=\left\{\begin{array}{ll}1, & m=0 \\ 2, & m \neq 0\end{array}\right.$ being the Neumann's symbol, and $P_{n}^{m}(\cdot)$ the associated Legendre function of the first kind and orders $n, m$. Then, the internal boundary-value problem for the Helmholtz equation (3.1) admits a classical solution

$$
\begin{equation*}
v(x, y, z) \in C^{2}(\mathcal{D}) \tag{3.7}
\end{equation*}
$$

such that the following spherical Bessel function expansion holds

$$
\begin{gather*}
v(\rho R(\vartheta, \varphi) \sin \vartheta \cos \varphi, \rho R(\vartheta, \varphi) \sin \vartheta \sin \varphi, \rho R(\vartheta, \varphi) \cos \vartheta) \\
=U(\rho, \vartheta, \varphi)=\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \jmath_{n}(k \rho R(\vartheta, \varphi))  \tag{3.8}\\
\times P_{n}^{m}(\cos \vartheta)\left(A_{n, m} \cos m \varphi+B_{n, m} \sin m \varphi\right) .
\end{gather*}
$$

For each pair of indices $n \in \mathbb{N}_{0}, m=0,1, \ldots, n$, define

$$
\begin{gather*}
{\left[\begin{array}{c}
\xi_{n, m}(\vartheta, \varphi) \\
\eta_{n, m}(\vartheta, \varphi)
\end{array}\right]=J_{n}(k \rho R(\vartheta, \varphi)) P_{n}^{m}(\cos \vartheta)\left\{\gamma\left[\begin{array}{l}
\cos m \varphi \\
\sin m \varphi
\end{array}\right]\right.} \\
+\frac{\lambda}{\sqrt{1+\psi(\vartheta, \varphi)^{2}}} \cdot\left[\begin{array}{cc}
\cos m \varphi & -\sin m \varphi \\
\sin m \varphi & \cos m \varphi
\end{array}\right]  \tag{3.9}\\
\left.\cdot\left[\begin{array}{c}
k \frac{j_{n}(k \rho R(\vartheta, \varphi))}{J_{n}(k \rho R(\vartheta, \varphi))}+\sin \vartheta \frac{\psi_{\varphi}(\vartheta, \varphi)}{R(\vartheta) \dot{P}^{m}(\cos \vartheta)} \\
-\frac{m}{\sin \vartheta} \frac{\psi_{\varphi}(\vartheta, \varphi), \varphi}{R(\vartheta, \varphi)}
\end{array}\right]\right\}
\end{gather*}
$$

where

$$
\begin{equation*}
j_{n}(z) \equiv \frac{d}{d z} \jmath_{n}(z)=-\jmath_{n+1}(z)+\frac{n}{z} \jmath_{n}(z), \tag{3.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{P}_{n}^{m}(z) \equiv \frac{d}{d z} P_{n}^{m}(z)=\frac{n z P_{n}^{m}(z)-(n+m) P_{n-1}^{m}(z)}{z^{2}-1} . \tag{3.11}
\end{equation*}
$$

Thus, the coefficients $A_{n, m}, B_{n, m}$ in (3.8) can be determined by solving the infinite linear system

$$
\sum_{n=0}^{+\infty} \sum_{m=0}^{n}\left[\begin{array}{cc}
\mathrm{X}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{+} & \mathrm{Y}_{\mathrm{q}}^{+}+\mathrm{p}, \mathrm{n}, \mathrm{~m}  \tag{3.12}\\
\mathrm{X}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}} & \mathrm{Y}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}
\end{array}\right] \cdot\left[\begin{array}{c}
A_{n, m} \\
B_{n, m}
\end{array}\right]=\left[\begin{array}{c}
\alpha_{q, p} \\
\beta_{q, p}
\end{array}\right]
$$

where

$$
\begin{align*}
& \mathrm{X}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{ \pm}=\epsilon_{\mathrm{p}} \frac{2 \mathrm{q}+1}{4 \pi} \frac{(\mathrm{q}-\mathrm{p})!}{(\mathrm{q}+\mathrm{p})!} \int_{0}^{2 \pi} \int_{0}^{\pi} \xi_{\mathrm{n}, \mathrm{~m}}(\vartheta, \varphi) \mathrm{P}_{\mathrm{q}}^{\mathrm{p}}(\cos \vartheta)  \tag{3.13}\\
& \times\left\{\begin{array}{c}
\cos p \varphi \\
\sin p \varphi
\end{array}\right\} \sin \vartheta d \vartheta d \varphi, \\
& \mathrm{Y}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{ \pm}=\epsilon_{\mathrm{p}} \frac{2 \mathrm{q}+1}{4 \pi} \frac{(\mathrm{q}-\mathrm{p})!}{(\mathrm{q}+\mathrm{p})!} \int_{0}^{2 \pi} \int_{0}^{\pi} \eta_{\mathrm{n}, \mathrm{~m}}(\vartheta, \varphi) \mathrm{P}_{\mathrm{q}}^{\mathrm{p}}(\cos \vartheta)  \tag{3.14}\\
& \times\left\{\begin{array}{c}
\cos p \varphi \\
\sin p \varphi
\end{array}\right\} \sin \vartheta d \vartheta d \varphi,
\end{align*}
$$

with $q \in \mathbb{N}_{0}, p=0,1, \ldots, q$.
Proof. Since the domain $\mathcal{D}$ becomes the unit sphere in the stretched co-ordinates system for the $x, y, z$ space, we can use the usual eigenfunction method [7] and separation of variables (with respect to the variables $r, \vartheta, \varphi)$. As a consequence, elementary solutions of the problem can be sought in the form

$$
\begin{equation*}
u(r, \vartheta, \varphi)=U\left(\frac{r}{R(\vartheta, \varphi)}, \vartheta, \varphi\right)=\mathrm{P}(\rho) \Theta(\vartheta) \Phi(\varphi) \tag{3.15}
\end{equation*}
$$

Upon substituting into the Helmholtz equation one can easily find that the functions $\mathrm{P}(\cdot), \Theta(\cdot), \Phi(\cdot)$ must satisfy the ordinary differential equations

$$
\begin{gather*}
r^{2} \frac{d^{2} \mathrm{P}(\mathrm{r})}{d r^{2}}+2 r \frac{d \mathrm{P}(\mathrm{r})}{d r}+\left(k^{2} r^{2}-\nu^{2}\right) \mathrm{P}(\mathrm{r})=0  \tag{3.16}\\
\frac{1}{\sin \vartheta} \frac{d}{d \vartheta}\left[\sin \vartheta \frac{d \Theta(\vartheta)}{d \vartheta}\right]+\left(\nu^{2}-\frac{\mu^{2}}{\sin ^{2} \vartheta}\right) \Theta(\vartheta)=0  \tag{3.17}\\
\frac{d^{2} \Phi(\varphi)}{d \varphi^{2}}+\mu^{2} \Phi(\varphi)=0 \tag{3.18}
\end{gather*}
$$

respectively. The parameters $\nu$ and $\mu$ are separation constants, whose choice is governed by the physical requirement that at any fixed point in space the scalar field $u(r, \vartheta, \varphi)$ must be single-valued. So, by setting

$$
\begin{gather*}
\mu=m \in \mathbb{Z},  \tag{3.19}\\
\nu^{2}=n(n+1) \quad\left(n \in \mathbb{N}_{0}\right), \tag{3.20}
\end{gather*}
$$

we find

$$
\begin{gather*}
\Phi(\varphi)=a_{m} \cos m \varphi+b_{m} \sin m \varphi  \tag{3.21}\\
\Theta(\vartheta)=c_{n, m} P_{n}^{m}(\cos \vartheta) \tag{3.22}
\end{gather*}
$$

where $a_{m}, b_{m}, c_{n, m} \in \mathbb{R}$ denote arbitrary parameters. To identify the radial function $\mathrm{P}(\cdot)$ satisfying (3.16), we can write

$$
\begin{equation*}
\mathrm{P}(\mathrm{r})=(\mathrm{kr})^{-\frac{1}{2}} \zeta(\mathrm{r}) . \tag{3.23}
\end{equation*}
$$

By doing so, it is readily shown that $\zeta(r)$ satisfies

$$
\begin{equation*}
r^{2} \frac{d^{2} \zeta(r)}{d r^{2}}+r \frac{d \zeta(r)}{d r}+\left[k^{2} r^{2}-\left(n+\frac{1}{2}\right)^{2}\right] \zeta(r)=0 \tag{3.24}
\end{equation*}
$$

and hence is a cylinder function of half order. In order to assure the boundedness of the solution, Bessel functions of the first kind must be considered for $\zeta(r)$, so that

$$
\begin{equation*}
\mathrm{P}(\mathrm{r})=\mathrm{d}_{\mathrm{n}, \jmath_{\mathrm{n}}}(\mathrm{kr}), \quad\left(\mathrm{d}_{\mathrm{n}} \in \mathbb{R}\right), \tag{3.25}
\end{equation*}
$$

where

$$
\begin{equation*}
\jmath_{n}(z)=\sqrt{\frac{\pi}{2 z}} J_{n+\frac{1}{2}}(z)=(2 z)^{n} \sum_{m=0}^{+\infty} \frac{(-1)^{m}(n+m)!}{m![2(n+m)+1]!} z^{2 m} \tag{3.26}
\end{equation*}
$$

denotes the spherical Bessel function of the first kind and order $n[1]$. Therefore, the general solution of the differential problem (3.1) can be sought in the form

$$
\begin{align*}
& u(r, \vartheta, \varphi)=\sum_{n=0}^{+\infty} \sum_{m=0}^{n} \jmath_{n}(k r) P_{n}^{m}(\cos \vartheta)  \tag{3.27}\\
& \quad \times\left(a_{n, m} \cos m \varphi+b_{n, m} \sin m \varphi\right) .
\end{align*}
$$

Enforcing the Robin boundary condition yields

$$
\begin{align*}
& F(\vartheta, \varphi)=\gamma u(R(\vartheta, \varphi), \vartheta, \varphi)+\lambda \frac{\partial u}{\partial \nu}(R(\vartheta, \varphi), \vartheta, \varphi)  \tag{3.28}\\
& =\gamma u(R(\vartheta, \varphi), \vartheta, \varphi)+\lambda \hat{\nu}(\vartheta, \varphi) \cdot \nabla u(R(\vartheta, \varphi), \vartheta, \varphi),
\end{align*}
$$

where

$$
\begin{equation*}
\nabla u(r, \vartheta, \varphi)=\hat{r} \frac{\partial u(r, \vartheta, \varphi)}{\partial r}+\hat{\vartheta} \frac{1}{r} \frac{\partial u(r, \vartheta, \varphi)}{\partial \vartheta}+\hat{\varphi} \frac{1}{r \sin \vartheta} \frac{\partial u(r, \vartheta, \varphi)}{\partial \varphi}, \tag{3.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\nu}(\vartheta, \varphi)=\frac{\hat{r}-\psi_{\vartheta}(\vartheta, \varphi) \hat{\vartheta}-\psi_{\varphi}(\vartheta, \varphi) \hat{\varphi}}{\sqrt{1+\psi(\vartheta, \varphi)^{2}}} \tag{3.30}
\end{equation*}
$$

Finally, by combining equations above and using the Fourier projection method, formulas (3.12)-(3.14) follow.

In a similar way, the external Robin problem

$$
\begin{cases}\Delta v(x, y, z)+k^{2} v(x, y, z)=0, & (x, y, z) \in \mathbb{R}^{3} \backslash \mathcal{D}  \tag{3.31}\\ \gamma v(x, y, z)-\lambda \frac{\partial v}{\partial \nu}(x, y, z)=f(x, y, z), & (x, y, z) \in \partial \mathcal{D}\end{cases}
$$

subject to the Sommerfeld radiation condition at infinity [14]

$$
\begin{equation*}
\lim _{r \rightarrow+\infty} r\left(\frac{\partial}{\partial r}-i k\right) v(x, y, z)=0 \tag{3.32}
\end{equation*}
$$

may be addressed. In particular, the following theorem can be easily proved.

Theorem 3.2 Under the hypotheses of the previous theorem, the external boundary-value problem for the Helmholtz equation (3.31)-(3.32) admits a classical solution

$$
\begin{equation*}
v(x, y, z) \in C^{2}\left(\mathbb{R}^{3} \backslash \mathcal{D}\right) \tag{3.33}
\end{equation*}
$$

such that the following series expansion holds

$$
\begin{gather*}
v(\rho R(\vartheta, \varphi) \sin \vartheta \cos \varphi, \rho R(\vartheta, \varphi) \sin \vartheta \sin \varphi, \rho R(\vartheta, \varphi) \cos \vartheta) \\
=U(\rho, \vartheta, \varphi)=\sum_{n=0}^{+\infty} \sum_{m=0}^{n} h_{n}^{(1)}(k \rho R(\vartheta, \varphi)) P_{n}^{m}(\cos \vartheta)  \tag{3.34}\\
\times\left(A_{n, m} \cos m \varphi+B_{n, m} \sin m \varphi\right)
\end{gather*}
$$

$h_{n}^{(1)}(\cdot)$ denoting the spherical Hankel function of the first kind and order $n$ [1]. For each pair of indices $n \in \mathbb{N}_{0}, m=0,1, \ldots, n$, define

$$
\begin{gather*}
{\left[\begin{array}{c}
\xi_{n, m}(\vartheta, \varphi) \\
\eta_{n, m}(\vartheta, \varphi)
\end{array}\right]=h_{n}^{(1)}(k \rho R(\vartheta, \varphi)) P_{n}^{m}(\cos \vartheta)\left\{\gamma\left[\begin{array}{l}
\cos m \varphi \\
\sin m \varphi
\end{array}\right]\right.} \\
-\frac{\lambda}{\sqrt{1+\psi(\vartheta, \varphi)^{2}}} \cdot\left[\begin{array}{cc}
\cos m \varphi & -\sin m \varphi \\
\sin m \varphi & \cos m \varphi
\end{array}\right]  \tag{3.35}\\
\left.\cdot\left[\begin{array}{c}
k \frac{\dot{h}_{n}^{(1)}(k \rho R(\vartheta, \varphi))}{h_{n}^{(1)}(k \rho R(\vartheta, \varphi))}+\sin \vartheta \frac{\psi_{\vartheta}(\vartheta, \varphi)}{R(\vartheta, \varphi)} \frac{\dot{P}_{n}^{m}(\cos \vartheta)}{P_{n}^{m}(\cos \vartheta)} \\
-\frac{m}{\sin \vartheta} \frac{\psi_{\varphi}(\vartheta, \varphi)}{R(\vartheta, \varphi)}
\end{array}\right]\right\},
\end{gather*}
$$

where

$$
\begin{equation*}
\dot{h}_{n}^{(1)}(z) \equiv \frac{d}{d z} h_{n}^{(1)}(z)=-h_{n+1}^{(1)}(z)+\frac{n}{z} h_{n}^{(1)}(z) . \tag{3.36}
\end{equation*}
$$

Thus, the coefficients $A_{n, m}, B_{n, m}$ in (3.34) can be determined by solving the infinite linear system

$$
\sum_{n=0}^{+\infty} \sum_{m=0}^{n}\left[\begin{array}{ll}
\mathrm{X}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{+} & \mathrm{Y}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{+}  \tag{3.37}\\
\mathrm{X}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{-} & \mathrm{Y}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}
\end{array}\right] \cdot\left[\begin{array}{c}
A_{n, m} \\
B_{n, m}
\end{array}\right]=\left[\begin{array}{c}
\alpha_{q, p} \\
\beta_{q, p}
\end{array}\right]
$$

where

$$
\begin{align*}
& \mathrm{X}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{ \pm}=\epsilon_{\mathrm{p}} \frac{2 \mathrm{q}+1}{4 \pi} \frac{(\mathrm{q}-\mathrm{p})!}{(\mathrm{q}+\mathrm{p})!} \int_{0}^{2 \pi} \int_{0}^{\pi} \xi_{\mathrm{n}, \mathrm{~m}}(\vartheta, \varphi) \\
& \times P_{q}^{p}(\cos \vartheta)\left\{\begin{array}{c}
\cos p \varphi \\
\sin p \varphi
\end{array}\right\} \sin \vartheta d \vartheta d \varphi,  \tag{3.38}\\
& \mathrm{Y}_{\mathrm{q}, \mathrm{p}, \mathrm{n}, \mathrm{~m}}^{ \pm}=\epsilon_{\mathrm{p}} \frac{2 \mathrm{q}+1}{4 \pi\left(\frac{(\mathrm{q}-\mathrm{p})!}{(\mathrm{q}+\mathrm{p})!}\right.} \int_{0}^{2 \pi} \int_{0}^{\pi} \eta_{\mathrm{n}, \mathrm{~m}}(\vartheta, \varphi)  \tag{3.39}\\
& \times P_{q}^{p}(\cos \vartheta)\left\{\begin{array}{c}
\cos p \varphi \\
\sin p \varphi
\end{array}\right\} \sin \vartheta d \vartheta d \varphi,
\end{align*}
$$

with $q \in \mathbb{N}_{0}, p=0,1, \ldots, q$.
Remark 2. Note that the above formulas still hold under the assumption that the function $R(\vartheta, \varphi)$ is a piecewise continuous function, and the boundary data are described by square integrable, not necessarily continuous, functions so that the relevant spherical harmonics coefficients $\alpha_{n, m}$, $\beta_{n, m}$ in equation (3.5) are finite quantities.

## 4 Numerical examples

In the following numerical examples, we assume for the boundary $\partial \mathcal{D}$ a general polar equation of the type

$$
\begin{equation*}
R(\vartheta, \varphi)=\left(\left|\frac{\sin \frac{p \vartheta}{2} \cos \frac{q \varphi}{4}}{\eta_{1}}\right|^{\nu_{1}}+\left|\frac{\sin \frac{p \vartheta}{2} \sin \frac{q \varphi}{4}}{\eta_{2}}\right|^{\nu_{2}}+\left|\frac{\cos \frac{p \vartheta}{2}}{\eta_{3}}\right|^{\nu_{3}}\right)^{-1 / \nu_{0}}, \tag{4.1}
\end{equation*}
$$

extending to the three-dimensional case the "superformula" introduced by J. Gielis [11]-[12].


Figure 1: Relative boundary error $e_{N}$ as function of the order $N$ of the expansion (4.3). The domain $\mathcal{D}$ is described by the polar equation (4.1) with $\eta_{1}=\eta_{2}=\eta_{3}=1$, $p=2, q=4, \nu_{0}=2, \nu_{1}=\nu_{2}=\nu_{3}=3$. The parameters appearing in the Robin boundary condition are selected to be $\gamma=4 / 5$ and $\lambda=1 / 5$, respectively.

Very different shapes of the considered domain, including ellipsoids, Lamé-type domains (also called Superellipsoids), ovaloids, $(p, q)$-fold symmetric figures can be obtained


Figure 2: Spatial distribution of the boundary values relevant to the spherical harmonic expansion $U_{N}(\rho, \vartheta, \varphi)$ of order $N=9$ approximating the solution of the internal and external Robin textitBVP in the domain $\mathcal{D}$ described by the polar equation (4.1) with $\eta_{1}=\eta_{2}=\eta_{3}=1, p=2, q=4, \nu_{0}=2, \nu_{1}=\nu_{2}=\nu_{3}=3$. The parameters appearing in

$$
\text { the Robin boundary condition are selected to be } \gamma=4 / 5 \text { and } \lambda=1 / 5 \text {, respectively. }
$$

by assuming suitable values of the parameters $p, q, \eta_{1}, \eta_{2}, \eta_{3}, \nu_{0}, \nu_{1}, \nu_{2}, \nu_{3}$ in (4.1). It is worth emphasizing that almost all three-dimensional normalpolar domains are described (or at least approximated in a close way) by the specified class of surfaces.

In order to assess the performance of the proposed algorithm in terms of numerical accuracy and convergence rate, the relative boundary error function has been defined as follows

$$
\begin{equation*}
e_{N}=\frac{\left\|\gamma U_{N}(1, \vartheta, \varphi) \pm \lambda \frac{\partial U_{N}}{\partial \nu}(1, \vartheta, \varphi)-F(\vartheta, \varphi)\right\|}{\|F(\vartheta, \varphi)\|} \tag{4.2}
\end{equation*}
$$

$\|\cdot\|$ denoting the usual $L^{2}(\partial \mathcal{D})$ norm, and $U_{N}(\rho, \vartheta, \varphi)$ the partial sum of order $N$ relevant to the spherical Bessel/Hankel function expansion approximating the solution of the specific Robin $B V P$ for the Helmholtz equation, namely

$$
\begin{gather*}
U_{N}(\rho, \vartheta, \varphi)=\sum_{n=0}^{N} \sum_{m=0}^{n} \Upsilon_{n}(k \rho R(\vartheta, \varphi))  \tag{4.3}\\
\times P_{n}^{m}(\cos \vartheta)\left(A_{n, m} \cos m \varphi+B_{n, m} \sin m \varphi\right),
\end{gather*}
$$

where $\Upsilon_{n}(\cdot)=\jmath_{n}(\cdot)$ for the internal problem, and $\Upsilon_{n}(\cdot)=h_{n}^{(1)}(\cdot)$ for the external one. Similarly, the sign of the nomal derivative term in (4.2) is to be selected according to the considered differential problem.

### 4.1 Robin problem in a cuboid-like domain

By assuming in (4.1) $\eta_{1}=\eta_{2}=\eta_{3}=1, p=2, q=4, \nu_{0}=2, \nu_{1}=\nu_{2}=$ $\nu_{3}=3$, the domain $\mathcal{D}$ features a cuboid-like shape. Let $f(x, y, z)=$ $z-7 y+2 i x$ be the function representing the boundary values. Then, the relative boundary error $e_{N}$ as function of the number $N$ of terms in the expansion (4.3) exhibits the behavior shown in Fig. 1. As it appears from Fig. 2, the selection of a reduced expansion order $N=9$ results in a resonably accurate Fourier-like representation of the solution.

Remark 3. If the boundary values have wide oscillations, a larger number $N$ of terms in the relevant spherical harmonic expansion is needed to achieve a reasonable numerical accuracy.

Remark 4. The $L^{2}$ norm of the difference between the exact solution and its approximate value is always vanishing in the interior (exterior) of the considered domain, and generally small on the boundary. Point-wise convergence seems to hold on the whole boundary, with the only exception
of a set of measure zero corresponding to the singular points for the function or its derivative. In these points, oscillations of the approximate solution, recalling the classical Gibbs phenomenon, usually appear.

## 5 Conclusion

The use of suitable stretched co-ordinate systems, reducing a starlike domain to a unit sphere, allows for the application of a Fourier-like projection method to the solution of a great variety of differential problems in complex three-dimensional domains. In this way, the adoption of cumbersome techniques such as finite-difference or finite-element methods can be avoided, and the analytical expression of the solution of different boundary-value problems is derived by using quadrature rules and solving reduced-order linear systems.

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## Conflict of Interests

The authors state that there are no conflicts of interest to be disclosed in the presented publication.

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