SAINT-VENANT'S PROBLEMS FOR THREE-LAYERED CONFOCAL ELLIPTIC TUBE WITH AN ORTHOTROPIC ELLIPTIC KERNEL

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Abstract

In this paper Saint-Venant's problem for composed confocal elliptic tube by means of the Faber's polynomial's is studied. Also a torsion problem when the tube is strengthened by the orthotropic elliptic kernel is investigated.

Key words and phrases: Saint-Venant's problem, Faber's polynomials, confocal ellipses, orthotropic kernel, torsion, bending

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1 Basic Equations

Some questions connected with Saint-Venant's problem are considered in the monographs of Love [2] and Muskhlishvili [3].

In the articles [4, 5] Saint-Venant's problem for two-layered elliptic tube and the problem of torsion are considered. The solution and the coefficients of the Faber's polynomials are found completely.

In this paper the torsion problem of three-layered isotropic confocal elliptic tube strengthened by the solid orthotropic elliptic kernel is studied.

Let us consider a cartesian coordinate system $Ox_1x_2x_3$ and the elliptic tube composed of three-layered confocal elliptic tubes of different elastic materials, occupying the domains Ω_1, Ω_2 and Ω_3, and bounded by the planes $x_3 = 0$ *and* $x_3 = l, (l > 0)$. We denote the lateral confocal elliptic surfaces by $\Gamma_0 - \Gamma_1, \Gamma_1 - \Gamma_2$ and $\Gamma_2 - \Gamma_3$ respectively. The equations of lateral surfaces $\Gamma_j (j = 0, 1, 2, 3)$ are written in the following form

$$f_0(x_1, x_2) = 0, \quad f_1(x_1, x_2) = 0, \quad f_2(x_1, x_2) = 0, \quad f_3(x_1, x_2) = 0. \qquad (1.1)$$

It is obvious that the normal cross-section of composed body occupying the domain $\Omega = \Omega_1 + \Omega_2 + \Omega_3$ will be the confocal elliptic domain $\omega = \omega_1 + \omega_2 + \omega_3$. Each domain ω_j will be a confocal elliptic ring bounded by confocal ellipses γ_j and γ_{j+1} with major and minor axis a_j, b_j and a_{j+1}, b_{j+1} (j=0,1,2), $a_k^2 - b_k^2 = c^2$, (k=0,1,2,3), where c is a focal length.

Hence, indicated composed elliptic tube has an elliptic hole bounded by elliptic surface Γ_0 with the exterior border Γ_3. For this body we will consider all Saint-Venant's problems, and also a torsion problem when the composed tube is strengthened by the solid elliptic domain Ω_0, occupied by the orthotropic material.

Thus, in our case we have a three-layered composed elliptic beam with the elliptic orthotropic kernel. It is proposed that the domains $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 are glued to each other without of split along the surfaces Γ_0, Γ_1 and Γ_2 respectively.

According to Hooke's law the components of stress,strains and displacements $\tau_{kl}^{(j)}, e_{kl}^{(j)}$, $u_k^{(j)}$ (k,l=1,2,3;j=0,1,2,3), in isotropic domains Ω_1, Ω_2 and Ω_3 will be given by the formula

$$\begin{aligned} &Ee_{11} = \tau_{11} - \nu(\tau_{33} + \tau_{22}), \quad Ee_{22} = \tau_{22} - \nu(\tau_{33} + \tau_{11}), \\ &Ee_{33} = \tau_{33} - \nu(\tau_{11} + \tau_{22}), \end{aligned} \tag{1.2}$$

where the components of strain have the following form

$$e_{jk} = D_j u_k + D_k u_j, \quad e_{jj} = D_j u_j, \quad (j, k = 1, 2, 3), \tag{1.3}$$

where $D_j = \partial/\partial x_j$.

For the orthotropic material, occupying the domain Ω_0, the Hooke's law may be written in the form [4]

$$\begin{aligned} &\tau_{jj} = A_{1j}e_{11} + A_{2j}e_{22} + A_{3j}e_{33} \ \ (j = 1, 2, 3), \tau_{12} = A_{66}e_{12}, \\ &\tau_{23} = A_{44}e_{23}, \quad \tau_{13} = A_{55}e_{13}, \end{aligned} \tag{1.4}$$

where A_{jk} are the elasticity constants. The strain components e_{11}, e_{22} and e_{33} may be written in the form

$$Ee_{jj} = \sigma_{j1}\tau_{11} + \sigma_{j2}\tau_{22} - \nu_j\tau_{33}, \quad (j = 1, 2), \quad Ee_{33} = \tau_{33} - \nu_1\tau_{11} - \nu_2\tau_{22}, \tag{1.5}$$

where E is the modulus of elasticity ν_1, ν_2 are the Poisson's ratios in the directions Ox_1, Ox_2.

Besides, in each domain Ω_j the equations of equilibrium are satisfied

$$D_1\tau_{1j} + D_2\tau_{2j} + D_3\tau_{3j} = 0, \quad (j = 1, 2, 3). \tag{1.6}$$

By $P(P_1, P_2, P_3)$ and $M(M_1, M_2, M_3)$ we denote the projections on axis Ox_j $(j = 1, 2, 3)$ of the resultant forces acting on the end $x_3 = h$ and the resultant moments respectively.

Components of the stresses τ_{jk} in each normal cross-section ω of the composed three-layered elliptic tube satisfy the following equations

$$\int\int_{\omega} \tau_{j3} d\omega = P_j, \quad \int\int_{\omega} x_2\tau_{33} - x_3\tau_{23} d\omega = M_1, \quad (j = 1, 2, 3)$$

$$\int\int_{\omega} x_3\tau_{13} - x_1\tau_{33} d\omega = M_2, \int\int_{\omega} x_1\tau_{23} - x_2\tau_{13} d\omega = M_3. \tag{1.7}$$

For the composed elliptic tube with the cross-section $\omega = \omega_1 + \omega_2 + \omega_3$, bounded by interior ellipse γ_0 and exterior ellipse γ_3 with interfaces γ_1 and γ_2 it is well-known the following formula

$$\int\int_{\omega} (\tau_{j3}) d\omega = \oint_{\gamma_0} x_j \tau_{n3} d\gamma + \oint_{\gamma_3} x_j \tau_{n3} d\gamma + \int\int_{\omega} x_j D_3 \tau_{33} d\omega$$

$$+ \sum_{k=1,2} \oint_{\gamma_k} x_j \{[\tau_{n3}]_k - [\tau_{n3}]_{k+1}\} d\gamma, \quad (k = 1, 2), \tag{1.8}$$

where

$$\tau_{nj} = \tau_{1j} n_1 + \tau_{2j} n_2 + \tau_{3j} n_3, \quad (j = 1, 2, 3),$$

In this case $n_3 = 0$,τnj are projections on the axis Ox_j of the stresses vector $\tau_n(\tau_{n1}+\tau_{n2}+\tau_{n3})$. The symbols $[...]_m$ and $[...]_{m+1}$ denote the limiting values of the expressions included in the brackets taken from the domains ω_m and ω_{m+1} respectively, where $n(n_1, n_2, n_3)$ is an outward normal to the domain ω_j.

It will be remarked that when the composed elliptic tube is strengthened by the solid orthotropic kernel, the expression $\oint_{\gamma_{0x_j}} \tau_{n3} d\gamma$ in the formula (1.8) will be substituted by expression $\oint_{\gamma_{0x_j}} \{[\tau_{n3}]_0 - [\tau_{n3}]_1\} d\gamma$. Besides the lateral surfaces Γ_0 and Γ_3 of three-layered elliptic tube must be free from acting of the exterior forces and also it is necessary to fulfill the conditions of the continuity of the vectors of the displacement and stresses, which cross the interfaces Γ_1 and Γ_2 from the adjoint domains. Since the equations of the indicated surfaces Γ_j are given by the equalities (1.1), these conditions take the following form $\tau_{nj} = 0, (j = 1, 2, 3)$, on the first and last surfaces

$$[u_j]_e = [u_j]_{e+1}, \quad [\tau_{nj}]_e = [\tau_{nj}]_{e+1}, \quad (j = 1, 2, 3; \quad e = 1, 2)$$

on the second and third surfaces at the interface between the domains Ω_l and Ω_{l+1}.

It is proposed that the different isotropic elastic materials composed of the three-layered tube and occupying the domains Ω_1, Ω_2 and Ω_3 have the identical Poisson's ratios, i.e. $\nu_1, \nu_2, \nu_3 = \nu$, but modulus of the elasticity E in the different domains Ω_j has the different values E_j. These restrictions do not spread on the torsion problem of composed beam with orthotropic kernel.

At last it will be remarked that it is proposed that the origin and axis Ox_1 and Ox_2 coincide with the generalized center of inertia and of principal axis of inertia, i.e. the following equalities are true

$$J_{jl} = 0 \ \ (j = 1, 2, 3),$$

where

$$J_{ji} = \int\int_{\omega} E x^{(j)} x^{(i)} d\omega,$$

$$x^{(1)} = x_1, ; x^{(2)} = x_2, \ \ x^{(3)} = 1;$$

$$J_{jj} > 0, \ \ (j, l = 1, 2, 3).$$

2 Extension by the Longitudinal Force and Bending Due to Couples of Forces

Let us consider the case when the force is applied on an upper base $x_3 = l$ of the three-layered isotropic elliptic tube which occupies the composed domain $\Omega = \Omega_1 + \Omega_2 + \Omega_3$. It is proposed that a system of the forces is statically equivalent to one force P_3 producing the extension acting parallel to the axis Ox_3 and two couples of forces producing the flexures of the tube in the planes Ox_2x_3 and Ox_1x_3, by the moments M_1 and M_2, respectively. Thus, in the conditions (1.7) we will take $P_1 = P_2 = M_3 = 0$. We seek the solution of the problem, as well as for the homogenous body, in components of the stresses and displacements in each domain Ω_k in the form

$$\tau_{j1} = \tau_{j2} = \tau_{j3} = 0, \ \ (\tau_{33})_k = E_k \sum_{e=1}^{3} C_e x^{(e)},$$

$$u_j = \sum_{e=1}^{3} C_e g_j^{(e)} - \frac{1}{2} C_j x_3^2, \ \ u_3 = x_3 \sum_{e=1}^{3} C_e x^{(e)}, \tag{2.1}$$

where

$$j = 1, 2; \ \ x^{(1)} = x_1, \ \ x^{(2)} = x_2, \ \ x^{(3)} = 1, \ \ 2g_J^{(j)} = (-1)^j \nu (x_2^2 - x_1^2),$$

$$g_j^{(3)} = \nu x_j, \ \ g_1^{(2)} = g_2^{(1)} = \nu x_1 x_2, \ \ (j = 1, 2). \tag{2.2}$$

It is known that the components of the outward normal $n(n_1, n_2)$ to the boundary γ of domain ω_γ satisfy the following equalities

$$n_\gamma (n_1 + i n_2)_\gamma = (\Theta_\gamma)^{-1} (b_{\gamma \cos\vartheta} + + i a_\gamma \sin\vartheta)^{0,5}, \tag{2.3}$$

where $0 \leq \vartheta \leq 2\pi$.

It is obvious that the expressions (2.1) satisfy the equations (1.7). After substitution of three first conditions from (2.1) into conditions (1.7) we get, that the first, second and sixth equations are satisfied identically. From the other equations of (1.7) the system of algebraic equations is obtained for the determination of coefficients C_e

$$C_1 J_{1j}^{(1)} + C_2 J_{2j}^{(1)} + C_3 J_{3j}^{(1)} = N_j, \quad (j = 1, 2, 3), \tag{2.4}$$

where

$$N_1 = -M_2, \quad N_2 = M_1, \quad N_3 = P_3. \tag{2.5}$$

Taking into the account (2.5) we get $J_{(jk)}^{(1)} = 0 \quad (j \neq k)$ and after simple calculations, the equalities (2.3) and (2.4) take the form

$$C_j = [J_{jj}^{(1)}]^{-1} N_j, \quad (j = 1, 2, 3),$$

where

$$4J_{11}^{(1)} = \pi[E_1(a_1^3 b_1 - a_0^3 b_0) + E_2(a_2^3 b_2 - a_1^3 b_1) + E_3(a_3^3 b_3 - a_2^3 b_2)],$$

$$4J_{(22)}^{(1)} = \pi[E_1(a_1 b_1^3 - a_0 b_0^3) + E_2(a_2 b_2^3 - a_1 b_1^3) + E_3(a_3 b_3^3 - a_2 b_2^3)],$$

$$J_{(33)} = \pi[E_1(a_1 b_1 - a_0 b_0) + +E_2(a_2 b_2 - a1 b_1) + E_3(a_3 b_3 - a_2 b_2)].$$

Thus, our problem is solved completely.

3 Bending of a Cantilever Under a Transverse Force

Let us consider the three-layered isotropic confocal elliptic tube, when external forces applied to the "upper" base $x_3 = l$ statically are equivalent to two bending forces P_1 and P_2 parallel to the axis Ox_1 and Ox_2 respectively and applied at the point $x_3^0(0, 0, l)$. Therefore, for the equilibrium of the part of the tube enclosed between the planes $x_3 = x_3^0$ and $x_3 = 0$, where $0 \leq x_3^0 < l$, it will be sufficient to require that the components of the stresses τ_{jk} in each cross-section of the tube satisfy the conditions (1.7), where

$$P_3 = M_1 = M_2 = M_3 = 0, \quad P_1 \neq 0, \quad P_2 \neq 0. \tag{3.1}$$

The solution of this problems are similar to the homogenous case, the components of the displacements u_j and stresses τ_{jk} in each domain Ω_k are found in the form

$$u_e = [(l - x_3) G_e g_e^{(e)} + \frac{1}{2} G_e x_3^2 + G_2 g_e^{(2)} - \frac{1}{2} G_e x_3^2 + (-1)^e G_0 x_{3-e} x_3], \quad (e = 1, 2),$$

$$u_3 = -[x_3(l - \frac{1}{2}x_3)(G_1x_1 + G_2x_2) - G_1F_1 - G_2F_2 + \frac{1}{3}(G_1x_1^3 + G_2x_2^3) - G_0F_0],$$

$$\tau_{e3} = \mu[D_e(G_1F_1 + G_2F_2 + G_0F_0) - G_ex_e^2 - G_1g_e^{(1)} - G_2g_e^{(2)} + (-1)^eG_0x_{3-e}],$$

$$\tau_{33} = E(x_3 - l)(G_1x_1 + G_2x_2), \tag{3.2}$$

where $e = 1, 2$. The expressions $g_k^{(\alpha)}$ are given by the equalities (3.2).

In each domain Ω_j the functions F_k and the constants G_k will be determined. Putting the expressions (3.2) into the equations of equilibrium (1.6) and boundary-contact conditions, we get that the functions F_k must be the solutions to the following boundary-contact equations

$$\triangle F_k^{(j)} = 0, \ \ (k = 0, 1, 2; \ \ j = 0, 1, 2), \tag{3.3}$$

in each domain $\omega_k(\Omega_k)$, (k=1,2,3),

$$[D_nF_k^{(j)}]_e = [H_k^{(j)}]_e, \ \ j = 1, 2, 3; \ \ k = 0, 1, 2, \tag{3.4}$$

on the interfaces $\gamma_1(\Gamma_1)$ and $\gamma_3(\Gamma_3)$,

$$\begin{aligned} &[\mu D_nF_k^{(j)}]_e - [\mu D_nF_k^{(j)}]_{e+1} = [\mu H_k^{(j)}]_e - [\mu H_k^{(j)}]_{e+1}, \\ &[F_k^{(j)}]_e = [F_k^{(j)}]_{e+1} \ \ (e = 1, 2) \end{aligned} \tag{3.5}$$

on the interfaces $\gamma_e(\Gamma_e)$, $(e = 1, 2)$, where

$$\triangle = D_1^2 + D_2^2, \ \ D_n = D_1n_1 + D_2n_2, \ \ D_j = \frac{\partial}{\partial x_j},$$

$$H_0 = x_2n_1 - x_1n_2, \ \ H_k = \frac{1}{2}[(2 + \nu)x_k^2 - \nu x_{3-k}^2]n_k + \nu x_1x_2n_{3-k}. \tag{3.6}$$

It must be remarked that, since in each domain $\Omega_k(\omega_k)$ $(k = 1, 2, 3)$ the Poisson's ratios are identical $(\nu_1 = \nu_2 = \nu_3 = \nu)$, the following equalities hold [1]

$$[g_k^{(j)}]_e = [g_k^{(j)}]_{e+1}, \ \ [H_k^{(j)}]_e = [H_k^{(j)}]_{e+1} \ \ (e = 1, 2), \tag{3.7}$$

on interfaces $\gamma_1(\Gamma_1)$ and$\gamma_2(\Gamma_2)$, where $\mu = E[2(1 + \nu)]^{-1}$ is modulus of the rigidity, ν is the Poisson's ratio and E is the modulus of elasticity.

In each domain $\omega_j(\Omega_j)$we seek the solution of (3.3),(3.4),(3.5),(3.6) in the following form

$$F_0^{(j)} = Re\Phi_0^{(j)}(z) = Re[m_2^{(j)}t_1^2 + +m_{(-2)}^{(j)}t_1^{-2}]_0,$$

$$F_k^{(j)} = Re\{-i)^{k-1}[m_1^{(j)}t1 + m_3^{(j)}t_1^3 + m_{-1}^{(j)}t_1^{-1} + m_{-3}^{(j)}t_1^{-3}]_k\}, \tag{3.8}$$

where

$$t_1 = z+w,\ \ z = x_1+ix_2,\ \ w = (z^2-c^2)^{0,5},\ \ a_j^2-b_j^2 = c^2\ \ (j=1,2,3)\ \ i^2=-1$$

$$(w)_{\gamma_e} = b_e \cos\vartheta + a_e \sin\vartheta,\ \ (t_1)_{\gamma_e} = p_e \exp(i\vartheta),$$
$$p_e = a_e + b_e,\ \ 0 < \vartheta \le 2\pi\ \ (e=0,1,2,3)$$

and a real constants $m_k^{k(j)}$ will be determined.

Also, the following equalities hold

$$D_1 t_1 = t_1 w^{-1},\ \ D_2 t_1 = it_1 w^{-1},$$

$$D_n F_k^{(j)} = Re[(n_1+in_2)_{\gamma_e}(\phi'^{(j)}_k)_{\gamma_e}],\ \ (e=0,1,2,3), \tag{3.9}$$

where

$$\{\phi'^{(j)}_k\}_{\gamma_e} = \{\frac{\partial}{z}\phi(z)\}_{\gamma_e} = [b_{\gamma_e}\cos\vartheta + a_\gamma \sin\vartheta)(m_1^{(j)} p \exp i\vartheta$$
$$-m_{-1}^{(j)} \exp -i\vartheta + m_3^{(j)} p_e^3 \exp 3i\vartheta - 3m_{-3}^{(j)} p_e^{-3} \exp -3i\vartheta].$$

After some simple transformations on the base of equalities (3.8)-(3.9), in each domain ω_j and on elliptic boundary γ_e for the functions $(F_k^{(j)})_{\gamma_e}$, $(D_n F_k^{(j)})_{\gamma_e}$ and expressions$(H_k)_{\gamma_e}$, we obtain

$$(F_k^{(j)}))_{\gamma_e} = p_e^{-3} Re[p_e^4 m_1(j)\exp(i\vartheta) + m_{-1}^{(j)}\exp(-i\vartheta) + p_e^6 m_3^{(j)}\exp(3i\vartheta)$$
$$+m_{-3}\exp(-3i\vartheta)]_k,$$
$$[D_n F_k^{(j)}]_{\gamma_e} = p_e^{-3}(a_e^2\sin^2\vartheta + b_e^2\cos^2\vartheta)^{-0,5}\Re[p_e^4 m_1^{(j)}\exp i\vartheta + 3m_3^{(j)} p_e^6 \exp 3i\vartheta$$
$$-p_e^2 m_{-1}^{(j)}\exp -i\vartheta - 3m_{-3}\exp -3i\vartheta]_k,$$

on γ_e

$$(H_k) = Re[A_1^{(k)}\exp i\vartheta + A_3^{(k)}\exp 3i\vartheta - A_{-1}^{(k)}\exp -i\vartheta - A_{-3}^{(k)}\exp -3i\vartheta]_{\gamma_e}, \tag{3.10}$$

where $j=1,2,3;\ \ k=1,2;\ \ e=0,1,2,3;$ On γ_e we have

$$A_1^{(1)} = A_{-1}^{(1)} = 6a_e^2 b_e + \nu b_e c^2,\ \ A_3^{(1)} = A_{-3}^{(1)} = (2+3\nu)a_e^2 b_e + \nu b_e^3;\ \ A_1^{(2)}$$
$$= A_{-1}^{(2)} = 6a_e b_e^2 + \nu a_e c^2;\ \ A_3^{(2)} = A_{-3}^{(2)} = (2+3\nu)a_e b_e^2 + \nu A_e^3.$$

Taking into the account the equalities (3.3)-(3.6),(3.9),(3.10), we can write the boundary conditions on the exterior boundaries γ_3 and γ_0 in the form

$$(D_n F_k(3))_3 \equiv p_3^{-3}\Theta_3^{-1} Re[p_3^4(m_1^{(3)})_k \exp i\vartheta + 3p_3^6(m_3^{(3)})_k \exp 3i\vartheta$$

$$-p_3^2(m_{-1}^{(3)})_k \exp -i\vartheta - 3m_{-3}^{(3)} \exp(-3i\vartheta)] = Re[A_1^{(3)} \exp(i\vartheta)$$

$$+A_3^{(3)} \exp(3i\vartheta) + A_{-1}^{(3)} \exp(-i\vartheta) + A_{-3}^{(3)} \exp(-3i\vartheta)]_3$$

$$(D_n F_k^{(1)})_0 \equiv p_0^{-3}\Theta_0^{-1}$$

$$Re[p_0^4(m_1^{(1)})_k \exp i\vartheta + 3p_0^6(m_3^{(1)})_k \exp 3i\vartheta - p_0^2(m_{-1}^{(1)})_k \exp -i\vartheta$$

$$-3(m_{-3}^{(1)})_k \exp -3i\vartheta = Re[(A_1^{(0)})_k \exp i\vartheta + (A_3^{(0)})_k \exp 3i\vartheta$$

$$-(A_{-1}^{(0)})_k \exp -i\vartheta - 3(A_{-3}^{(0)})_k \exp -3i\vartheta]_0. \tag{3.11}$$

Taking into the account the boundary-contact conditions we obtain

$$p_1^{(-3)}\{[F_k^{(1)}]_1 - [F_k^{(2)}]_2\} \equiv Re[|p_1^4|(m_1^{(1)}) \exp i\vartheta + p_1^2(m_{-1}^{(1)}) \exp -i\vartheta$$

$$p_1^6(m_3^{(1)}) exp 3i\vartheta + (m_{-3}^{(1)}) \exp -3i\vartheta]_k - Re[p_1^4(m_1^{(2)}) \exp i\vartheta$$

$$+p_1^2(m_{-1}^{(2)}) \exp -i\vartheta + p_1^6(m_3^{(2)}) \exp 3i\vartheta + (m_{-3}^{(2)}) \exp -3\vartheta i]_k = 0'$$

$$p_2^{(-3)}\{[F_k^{(2)}]_2 - [F_k^{(3)}]_3\} \equiv Re[p_2^4(m_1^{(2)}) \exp i\vartheta + p_2^2(m_{-1}^{(2)}) \exp -i\vartheta]$$

$$+[p_2^6(m_3^{(2)}) \exp(3i\vartheta)][+(m_{-3}^{(2)}) \exp -3i\vartheta]_k - Re[p_2^4(m_1^{(3)}) \exp i\vartheta + p_2^2(m_{-1}^{(3)})$$

$$\exp -i\vartheta + p_2^6(m_3^{(3)}) \exp 3i\vartheta + (m_{-3}^{(3)}) \exp -3i\vartheta]_k = 0,$$

$$\mu_1[D_n F_k^{(1)}]_1 - \mu_2[D_n F_k^{(2)}]_2 = p_1^{-3}\Theta_1^{-1}\{\mu_1 Re[(p_1^4 m_1^{(1)}) \exp i\vartheta$$

$$+3p_1^6(m_3^{(1)}) \exp 3i\vartheta - p_1^2(m_{-1}^{(1)}) \exp -i\vartheta - 3(m_{-3}^{(1)}) \exp -3i\vartheta]_k - \mu_2 Re[p_1^4(m_1^{(2)})$$

$$\exp i\vartheta + 3p_1^6(m_3^{(2)}) \exp 3i\vartheta - p_1^2(m_{-1}^{(2)}) \exp -i\vartheta - 3(m_{-3}^{(2)}) \exp -3i\vartheta]_k\}$$

$$= [(\mu_1) - (\mu_2)]((H_k))_1, \tag{3.12a}$$

$$\mu_2[D_n F_k^{(2)}]_2 - \mu_3[[D_n F_k^{(3)}]_3 = [\Theta_2^{-1}](p_2^{-3})\{\mu_2 Re[p_2^4(m_1^{(2)}) \exp i\vartheta$$

$$+3p_2^6(m_3^{(2)}) \exp 3i\vartheta - p_2^2(m_{-1}^{(2)}) \exp -i\vartheta - 3(m_{-3)}^{(2)}) \exp -3i\vartheta$$

$$-\mu_3 Re[p_2^4(m_1^{(3)}) \exp i\vartheta + 3p_2^6(m_3^{(3)}) \exp 3i\vartheta - p_2^2(m_{-1}^{(3)}) \exp -i\vartheta$$

$$-3(m_{-3}^{(3)}) \exp -3i\vartheta]\} = (\mu_2 - \mu_3)(H_k)_2, \tag{3.12b}$$

where the functions $(H_k)_{\gamma_e}$ are given by the equalities (3.10).

Equating multipliers at the same powers of $\exp \pm\vartheta$ in the equalities (12a),(12b) for the determination of coefficients $(m_l^{(j)})_k$ and $(m_{-l}^{(j)})_k$ we get the following linear algebraic equations

$$(m_k^{(1)}\rho_0^{2k} - m_{-k}^{(1)})_0 = \frac{1}{k^1}(A_k^{(1)})_0\rho_0^k,$$

$$(m_k^{(3)}\rho_3^{2k} - m_{-k}^{(3)})_{\gamma_3} = \frac{1}{k^1}(A_k^{(3)})_3\rho_3^k, (k = 1.3)$$

$$\mu_1[m_k^{(1)}\rho_1^{2k} - m_{-k}^{(1)}] - \mu_2[m_k^{(2)}(\rho_1)^{2k} - m_{-k}^{(2)}]$$

$$= \mu_1\frac{1}{k^1}(A_k(1))_1\rho_1^k - \mu_2(A_k^{(2)})_1, \ \ (k = 1, 3),$$

$$(m_k^{(1)})\rho_1^{2k} + (m_{-k}^{(1)}) - ((m_k^{(2)})\rho_1^{2k} + m_{-k}^{(2)}) = 0;$$

$$\mu_2[m_k^{(2)}\rho_2^{-}m{=}\, k^{(2)}] - \mu_3[m_k^{(3)}\rho_2^{2k} - m_{-k}^{(3)}]$$

$$= \mu_2\frac{1}{k^1}(A_k^{(2)})_2\rho_2^k - \mu_3(A_k^{(3)})_2\rho_2^k,$$

$$m_k(2)\rho_2^{(2k)} + m_{-k}^{(2)} - m_k^{(3)}\rho_2^{2k} - m_{-k}^{(3)} = 0, \ \ (k = 1, 3). \tag{3.13}$$

From the first four equations of these expressions we obtain

$$m_{-k}^{(1)} = m_k(1)\rho_0^{2k} - (A_k^{(1)})_{\gamma_0}\rho_0^k, m_{-k}^{(3)}$$

$$= m_k^{(3)}\rho_3^{2k} - (A_k^{(3)})_{\gamma_3}\rho_3^k, \ \ (k = 1, 3),$$

Substituting these meanings into other corresponding equations from (3.13) we get

$$\mu_1 m_k^{(1)}(\rho_1^{2k} - \rho_0^{2k}) - \mu_2(m_k^{(2)}\rho_1^{2k} - m_{-k})$$

$$= \mu_1\frac{1}{k^1}[A_k^{(1)}\rho_1^k - A_k^{(1)}\rho_0^k] - \mu_2 A_k^{(2)}\rho_1^k,$$

$$m_k^{(1)}(\rho_1^{2k} + \rho_0^{2k}) - m_k(2)\rho_1^{2k} - m_k^{(2)}\rho_1^{2k} - m_{-k}^{(2)} = \frac{1}{k^1}(A_k^{(1)})_{\gamma_0}\rho_0^{2k}, \tag{3.14}$$

$$\mu_2(m_k^{(2)}\rho_2^{2k} - m_{-k}^{(2)}) - \mu_3 m_k^{(3)}(\rho_2^{2k} - \rho_3^{2k} == \frac{1}{k^1}\mu_2(A_k^{(2)})\gamma_2\rho_2^k -$$

$$\frac{1}{k^1}\mu_3[A_k^{(3)}\rho_2^{2k} - A_k^{(3)}\rho_3^{2k}],$$

$$m_k^{(2)}\rho_2^{2k} + m_{-k}^{(2)} - m_k^{(3)}(\rho_2^{2k} + \rho_3^{2k}) = \frac{1}{k^1}(A_k^{(3)})\rho_3^k. \tag{3.15}$$

In (3.14) multiply for k=1 the first equation by $\rho_1^2 + \rho_0^2$ and from obtained one subtract the third equation multiplied by $\mu_1(\rho_1^2 - \rho_0^2)$, then multiply for k=3 the second equation by $\rho_1^6 + \rho_0^6$ and from obtained one subtract the fourth equation multiplied by $\mu_1(\rho_1^6 - \rho_0^6)$, we get

$$L_1 = \mu_1(\rho_1^2 - \rho_0^2)\{(m_1^{(2)}\rho_1^2 + m_{-1}^{(2)}) - \mu_2(\rho_1^2 + \rho_0^2)(m_1(2)\rho_1^2 - m_{-1}^{(2)})\},$$

$$L_2 = \mu_1(\rho_1^6 - \rho_0^6)\{(m_3^{(2)}\rho_1^6 + m_{-3}^{(2)}) - \mu_2(\rho_1^6 + \rho_0^6)(m_3^{(2)}\rho_1^6 - (m^{(2)})_{-3})\}, \tag{3.16}$$

where

$$L_1 = (\rho_1^2 + \rho_0^2)\{\mu_1[A_1^{(1)}(\rho_1 - \rho_0)] - \mu_1(\rho_1^2 - \rho_0^2)A_1^{(1)},$$

$$L_2 = (\rho_1^6 + \rho_0^6)\{(\mu_1 \frac{1}{3}[A_3^{(1)}(\rho_1^3 - \rho_0^3)] - \mu_2(A_3^{(2)}\rho_1^3\}$$

$$-\mu_1(\rho_1^6 - \rho_0^6)\frac{1}{3}A_3^{(1)}\rho_0^3;$$

Analogously for the equations (3.15). In (3.15) multiply for k=1 the first equation by $\rho_2^2 + \rho_3^2$ and from obtained one subtract the third equation multiplied by $\mu_3(\rho_2^2 - \rho_3^2)$, then multiply for k=3 the second equation by $\rho_2^6 + \rho_3^6$ and from obtained one subtract the fourth equation multiplied by $\mu_3(\rho_2^6 - \rho_3^6)$, we get

$$\mu_2(\rho_2^2 + \rho_3^2)(m_1^{(2)}\rho_2^2 - m_{-1}^{(2)}) - \mu_3(\rho_2^2 - \rho_3^2)(m_1^{(2)}\rho_2^2 + m{-}1^{(2)} = L_3,$$

$$\mu_2(\rho_2^6 + \rho_3^6)(m_3^{(2)}\rho_2^6 - m{-}3^{(2)}) - \mu_3(\rho_2^6 - \rho_3^6)(m_3^{(2)}\rho_2^6 + m{-}3^{(2)}) = L_4, \quad (3.17)$$

where

$$L_3 = (\rho_2^2 + \rho_3^2)\{\mu_2(A_1^{(2)})\rho_2 - \mu_3[A_1^{(3)}(\rho_2 - \rho_3)] - \mu_3(\rho_2^2 - \rho_3^2)A_1^{(3)}\rho_3)\},$$

$$L_4 = (\rho_2^6 + \rho_3^6)\frac{1}{3}\{\mu_2 A_3(2)\rho^3 - \mu_3[A_3(3)(\rho_2^3 - \rho_3^3)] - \mu_3\frac{1}{3}(\rho_2^6 - \rho_3^6)A_3(3)\rho_3^3),$$

$$\rho_k^j = (a_k + b_k)^j.$$

From (3.16),(3.17) we get

$$\rho_1^2[\mu_2(\rho_1^2 + \rho_0^2)) - \mu_1(\rho_1^2 - \rho_0^2)]m_1^{(2))} - [\mu_2(\rho_1^2 + \rho_0^2) + \mu_1(\rho_1^2 - \rho_0^2]m_{-1}^{(2)} = -L_1,$$

$$\rho_2^2[\mu_3(\rho_2^2 - \rho_3^2) - \mu_2(\rho_2^2 + \rho_3^2)]m_1^{(2)} + [\mu_3(\rho_2^2 - \rho_3^2) + \mu_2(\rho_2^2 + \rho_3^2]m_{-1}^{(2)}$$
$$= -L_3; \quad (3.18)$$

$$\rho_1^6[\mu_2(\rho_1^6 + \rho_0^6) - \mu_1(\rho_1^6 - \rho_0^6)]m_3(2) - [\mu_1(\rho_1^6 - \rho_0^6) + \mu_2(\rho_1^6 + \rho_0^6)]m_{-3}(2) = -L_2,$$

$$-\rho_2^6[\mu_2(\rho_3^6 + \rho_2^6) + \mu_3(\rho_3^6 - \rho_2^6)]m_3(2) + [\mu_2(\rho_2^6 + \rho_3^6) - \mu_3(\rho_3^6 - \rho_2^6)]m_{-3}2$$
$$= -L_4. \quad (3.19)$$

It is easy to show ,that the determinants of the system (3.18), (3.19) $\nabla_1 > 0$ and $\nabla_2 > 0$ are given by the equalities

$$\nabla_1 = \mu_1\mu_2(\rho_2^2 + \rho_3^2)(\rho_1^2 - \rho_0^2)(\rho_2^2 + \rho_1^2) + + mu_1\mu_3(\rho_1^2 - rho_0^2)(\rho_2^2 - \rho_1^2)(\rho_3^2 - \rho_2^2) +$$

$$\nabla_2 = \mu_1\mu_2(\rho_2^6 + \rho_3^6)(\rho_1^6 - \rho_0^6)(\rho_2^6 - \rho_1^6) +$$
$$+\mu_1\mu_3(\rho_1^6 - \rho_0^6)(\rho_2^6 + \rho_1^6)(\rho_3^6 + \rho_2^6) +$$
$$+\mu_2^2(\rho_1^6 + \rho_0^6)(\rho_2^6 + \rho_3^6)(\rho_2^6 + \rho_1^6) + mu_3(\rho_1^6 + \rho_0^6)(\rho_3^6 - \rho_2^6)(\rho_2^6 - \rho_1^6).$$

Thus, the problem is solved completely.

4 On the Torsion of Three Layered Isotropic Elliptic Beam Strengthened by the Orthotropic Solid Elliptic Beam

We consider a composed isotropic three-layered confocal elliptic cylindrical beam (see paragraph 1). In the problem of torsion we must carry out the following "end" conditions on cross-section $x_3 = const$

$$\int\int_{\omega} \tau_{j3} d\omega = 0, \ (j = 1, 2, 3),$$

$$\int\int_{\omega} (x_2\tau_{33} - x_3\tau_{23}) d\omega = \int\int_{\omega} \tau_{33} d\omega = 0,$$

$$\int\int_{\omega} (x_1\tau_{23} - x_2\tau_{13}) d\omega = M_3, \tag{4.1}$$

where M_1 is the given twisting moment, $\omega = \omega_0 + \omega_1 + \omega_3$, ω_1, ω_2 and ω_3 are domains occupied by the isotropic materials and ω_0 is domain occupied by the orthotropic material. We seek the solution of the torsion problem in isotropic and orthotropic domains in the form

$$u_1^{(j)} = -Gx_2x_3,$$

$$u_2^{(j))} = Gx_1x_3,$$

$$u_3^{(j)} = Gf_j(x_1, x_2),$$

where the constant G and the functions f_j will be determined. By the formula (1.3)-(1.7)the components of the stress for the displacement in different isotropic domains ω_j (j=1,2,3) are

$$\tau_{13}^{(j)} = G\mu_j(D_1 f_j - x_2), ; \ \tau_{23}^{(j)} = G\mu_j(D_2 f_j + x_1); (j = 1, 2, 3), \tag{4.2}$$

in the orthotropic domain ω_0

$$\tau_{13}^{(j)} = GA_{55}(D_1 f_0 - x_2), \ \tau_{23}^{(j)} = GA_{44}(D_2 f^{(0)} + x_1); \ A_{44}A_{55} > 0, \tag{4.3}$$

where μ_j is the shear modulus and A_{44} and A_{55} are modulus of the orthotropic material.

Substituting expressions (4.3) and (4.4) in the equations of elastic equilibrium (1.7) we obtain, that the functions f_1, f_2 and f_3 will be solutions of the Laplace equation in domains ω_1, ω_2 and ω_3 respectively

$$\triangle f_j = 0, (j = 1, 2, 3). \tag{4.4}$$

In the domain ω_0 the function f_0 satisfies the equation

$$A_{55}D_1^2 f_0 + A_{44}D_2^2 f_0 = 0, \tag{4.5}$$

where A_{44} and A_{55} are the elastic constants (shear modulus) of orthotropic material and $\triangle = D_1^2 + D_2^2$. The characteristic root of the equation (4.6) is

$$v_\star = i\sqrt{A_{55}A_{44}^{(-1)}},\ (i^2 = -1). \tag{4.6}$$

Let us introduce the complex variables

$$z = x_1 + ix_2;\ z_\star = x_1 + v_\star x_2,;\ t = z + w,;\ t_{\star 1} = \eta_\star^{(-1)}[z_\star + w_\star];$$

$$t_{\star 2} = \eta_{(\star)}^{(-1)}[z_* - w_\star]\eta_\star = (a_0 + b_0\sqrt{A_{(55)}A_{(44)}^{(-1)}});$$

$$w = \sqrt{z^2 - c^2}, w_\star = \sqrt{z_\star^2 - a_0^2 - v_\star^2 b_0^2}, \tag{4.7}$$

where $a_j^2 - b^2 = c^2,;\ (j = 0, 1, 2, 3)$ and $v_\star$ are given by the equality (4.7).

We seek the functions f_j in the form

$$f_j = 2^{-1}c^2 Re[i(m_j t^2 + m_j t^{-2})],;\ (j = 1, 2, 3) \tag{4.8}$$

$$f_0 = 2^{-1}c^2 Re[im_0(t_{\star 1}^2 + t_{\star 2}^2)]. \tag{4.9}$$

It will be noted that all functions are single-valued.

After simple transformations from (4.9) we obtain

$$D_1(t^2) = 2t^2 w^{(-1)},;\ D_2(t^2) = 2it^2 w^{(-1)};;\ \ D_1(t_{\star 1}^2) = 2(w_{(\star)})^{(-1)} t_\star^2$$

$$D_2(t_\star^2 1) = 2i\sqrt{A_{(55)}A_{(44)}^{(-1)}}$$

$$(w_{(\star)})^{(-1)} t_\star^2, -kD_1(t_{\star 2}^2) = -2(w_(\star))^{(-1)} t_{(\star)}^2, D_2(t_{(\star 2)}^2)$$

$$= -2i\sqrt{A_{(55)}A_{(44)}^{(-1)}}(w_{(\star)})^{(-1)} t_{\star 2}^2. \tag{4.10}$$

At the elliptic boundary $\gamma_e(x_1 = a_e \cos\vartheta,;\ \ x_2 = b_e \sin\vartheta)$ of the domain ω_e the following formulas are true

$$[t_{\star 1}]_{\gamma_0} = \exp(i\vartheta),$$

$$[t_{(\star 2)}]_{(\gamma_0)} = \lambda_0 \exp(-i\vartheta),$$

$$(w_0)_\gamma == ia_0 \sin\vartheta + b_0\sqrt{A_{55}}A_{44}^{(-1)} \cos\vartheta,\ \ (w)_{\gamma_e} = ia_e \sin\vartheta + b_e \cos\vartheta.$$

$$(t)_{\gamma_e} = p_e \exp(i\vartheta),$$

$$(n_1 + in_2)_{\gamma_e} = (\Theta^{(-1)} ia_e \sin\vartheta),$$
$$(\Theta)_{\gamma_e} \sqrt{a_e^2 \sin^2\vartheta + b_e^2 \cos^2\vartheta},$$
$$\lambda_0 = (a_0 - b_0\sqrt{A_{55}A^{(-1)44}}),$$
$$(a_0 + b_0\sqrt{A_{55}A^{-1}_{(44)}})^{-1}, \tag{4.11}$$

where

$$p_e = a_e + b_e,\ (0 < \vartheta \le 2\pi).$$

After simple calculations we obtain

$$D_1 f_j = Re[i(w)_j^{-1}(m_j t^2 - m{-}jt^{-2})],$$
$$D_2 f_j = -Re[(w)_j^{-1}(m_j t^2 - m_{-j}t^{-2}],$$
$$D_1 f_0 = Re[im_0 w_0^{-1}(t_{\star 1}^2 - t_{\star 2}^2)],$$
$$D_2 f_0 = -\sqrt{A_{55}A^{-1}}_{44} Re[m_0 w_0^{-1}(t_{\star 1}^2) - t_{\star 2}^2]. \tag{4.12}$$

Taking into the account equalities (4.11)-(4.12) we get

$$(D_n f_j)_{\gamma_k} \equiv (n_1 D_1 f_j + n_2 D_2 f_j)_{\gamma_k} =$$
$$= Re[-c^2(\Theta_k p_k)^{(} - 1)(m^{(}j)p^4 + m^{(}j)_{-1})_k] \sin 2\vartheta;$$
$$(f_j)_{\gamma_k} = -Re[c^2(2p_k)^{-1}(m_1^{(j)}p^4 - m_k^{(j)})] \sin 2\vartheta,$$
$$(D_n^\star)_\gamma 0 \equiv [A_{55}n_1 D_1 f_0 + A_{44}n_2 D_2 f_0]_{\gamma_0} =$$
$$= -c^2\sqrt{A_{44}A_{55}}\Theta_0^{-1}(1+\lambda_0)m_0 \sin 2\vartheta, (f_0)_{\gamma_0} = -c^2 m_0(1-\lambda_0)\sin 2\vartheta. \tag{4.13}$$

Substituting expressions (4.12) in the boundary-contact conditions for the determination of the real coefficients m_j the following linear algebraic equations are obtained

$$p_0^2\sqrt{A_{44}A_{55}}(1+\lambda_0)m_0 - \mu_1(m_1 p_0^4) + m_{-1}) =$$
$$= (2^{-1}p_0^2[\mu_1 - (A_{55}b_0^2 - A_{44}a_0^2)],$$
$$m_0 p_0^2(1-\lambda_0) - (m_1 p_0^4 - m_{-1}) = 0,$$
$$\mu_1(m_1 p_1^4 + m_{-1}) - \mu_2(m_2 p_1^4 + m_{-2}) = 2^{-1}p_1^2(\mu_2 - \mu_1),$$
$$m_1 p_1^4 - m_{-1} - m_2 p_1^4 + m_{-2} = 0,$$
$$\mu_2(m_2 p_2^4 + m_{-2}) - \mu_3(m_3 p_2^4 + m_{-3}) = 2^{-1}p_2^2(\mu_2 - \mu_3),$$
$$m_2 p_2^4 - m_{-2} - m_3 p^4 + m_{-3} = 0,$$

$$m_3p_3^4 + m_{-3} = 2^{-1}p_3^2. \tag{4.14}$$

From (4.14) we have

$$m_{-3} = m_3p_3^4 - 2^{-1}p_3^2.$$

Using the following formula $\Theta^{\pm}_{jk} \equiv (p_j^4 pm p_k^4)$ in the fifth and sixth equations of (4.14) one obtains

$$\mu_2(m_2p_2^4 + m_{-2}) - \mu_3m_3\Theta^+_{32} = 2^{-1}[p_2^2(\mu_2 - \mu_3) - \mu_3p_3^2],$$

$$m_2p_2^4 - m_{-2} + m_3\Theta^-_{32} = 2^{-1}p_3^2. \tag{15$*$}$$

From the last equation of (15∗) we have

$$m_{-2} = m_2p^4 + m_3\Theta^-_{32} - 2^{-1}p_3^2.$$

Taking into the account this formula in (4.13), (4.14) we can write

$$p_0^2\sqrt{A_{44}A_{55}}(1+\lambda_0)m_0 - \mu_1(m_1p_0^2 + m_{-1}) = 2^{-1}p_0^2[\mu_1 - (A_{55}b_0^2 - A_{44}a_0^2)],$$

$$p_0^2(1-\lambda_0)m_0 - m_1p_0^4 + m_{-1} = 0,$$

$$\mu_1(m_1p_1^4 + m_{-1}) - \mu_2[m_2\Theta^+_{21} - m_3\Theta^-_{32} = 2^{-1}[\mu_2p_3^2 + p_1^2(\mu_1 - \mu_2)],$$

$$m_1p^41 - m_{-1} + m_3\Theta^-_{32} + m_2\Theta^-_{21} = 2^{-1}p_3^2,$$

$$2p_2^4\mu_2m_2 + \mu_2m_3\Theta^-_{32} - \mu_3m_3\Theta^+_{32} = 2^{-1}(\mu_2 - \mu_3)\Theta^-_{32}. \tag{4.15}$$

Now we must eliminate coefficient m_{-1} from equations (4.15). After some transformations we obtain

$$m_0p_0^2[\mu_1(1-\lambda_0) + \sqrt{A_{44}A_{55}}(1+\lambda_0)] + 2\mu_1p_0^4m_1 = 2_{-1}[\mu_1 - (A_{55}b_0^2 - A_{44}a_0^2)],$$

$$\mu_1p_0^2(1-\lambda_0) - \mu_1m_1\Theta^+_{10} + \mu_2m_2\Theta^+_{21} + \mu_2m_3\Theta^-_{32} = 2^{-1}[p_1^2(\mu_2 - \mu_1) - \mu_2p_3^2],$$

$$m_1\Theta^-_{10} + p_0^2m_0(1-\lambda_0) + m_2\Theta^-_{21} + \mu_3\Theta^-_{32} = 2^{-1}p_3^2,$$

$$2\mu_2p_2^4m_2 + \mu_2m_3\Theta^-_{32} - \mu_3m_3\Theta^+_{32}\Theta^+_{32} = 2^{-1}(\mu_2 - \mu_3)(p_3^2 + p_2^2). \tag{4.16}$$

Now we will eliminate coefficient m_3 from (4.16). From the third equation of (4.16) the following relations are obtained

$$m_3 = (\Theta^-_{21})^{-1}[2^{-1}p_3^2 - m_0p_0^2(1-\lambda_0) - m_1\Theta^-_{10} - m_2\Theta^{21}.$$

Substituting this expression in other equations of (4.16) we obtain

$$m_0p_0^2[\mu_1(1-\lambda_0) + (1+\lambda_0)\sqrt{A_{44}A_{55}})] - 2\mu_1p_0^2m_1 = 2^{-1}p_0^2[\mu_1 - (A_{55}b_0^2 - A_{44}a_?^2],$$

$$(\mu_2\Theta^-_{10} + \mu_1\Theta^+_{10})m_1 - 2\mu_2p_1^4m_2 + p_0^2(\mu_2 - \mu_1)(1-\lambda_0)m_0 = 2^{-1}p_3^2(\mu_2 + \mu_3),$$

$$-mu_2m_2\Theta^-_{32}[2p_1^4 + \Theta^-_{21} + [\mu_3\Theta^+_{32}\mu_2\Theta^-_{32}][m_1\Theta^-_{10} + m_0p_0^2(1-\lambda_0)]$$

$$= 2^{-1}(\mu_2 - \mu_3)\Theta^+_{32}\Theta^-_{32} + +2^- p_3^2(\mu_3\Theta^+_{32} - \mu_2\Theta^-_{32}). \qquad (4.17)$$

Now we will eliminate m_2 from (4.17). From the second equation of (4.17) we obtain

$$m_2 = (2\mu_1 p_1^4)^{-1}[m_1(\mu_2\Theta^-_{10}+\mu_1\Theta^+_{10}++p_0^2(\mu_2-\mu_1)(1-\lambda_)m_0-2^{-1}p_3^2(\mu_2+\mu_3)].$$

Taking into the account this expression the system (4.17) takes the form

$$m_0\mu_2 p_0^2(1-\lambda_0)[2p_1^4(\mu_3\Theta^+_{32} - \mu_2\Theta^-32)] - (\mu_2-\mu_1)\Theta^-_{32}\Theta^+_{21}] + m_1$$
$$[2\mu_2 p_1^4\Theta^-_{10}(\mu_3\Theta^+_{32} - \mu_2\Theta^-_{32}) - \mu_2\Theta^-_{32}\Theta^+21(\mu_2\Theta^-_{10}\mu_1 Theta^+_{10}]$$
$$= 2^{-1}[\Theta^-_{32}(\mu_2-\mu_1)(p_3^2+p_2^2) + p_3^2(\mu_3\Theta +_{32} - \mu_2 Theta^-_{32}],$$
$$m_0[\mu_1(1-\lambda_0) + (1+\lambda_0)\sqrt{A_{44}A_{55}}] - 2\mu_1 p_0^2 m_1$$
$$= 2^{-1}(\mu_1 - A_{55}b_0^2 + A_{44}a_0^2). \qquad (4.18)$$

The determinant of the system (4.18) will be

$$\nabla = [\mu_1(1-\lambda_0) + (1+\lambda_0)\sqrt{A_{44}A_{55}}][2\mu_2 p_1^4\Theta^-_{10}(\mu_3\Theta^+_{32} - \mu_2\Theta^-_{32})$$
$$-\mu_2\Theta^-_{32}\Theta^+_{21}(\mu_2\Theta^-_{10} + \mu_1\Theta^+_{10}] + 2\mu_1 p_0^4\mu_2(1-\lambda_0)[2p_1^4(\mu_3\Theta^+_{32} - \mu_2\Theta^-_{32})$$
$$-\Theta^-_{32}\Theta^+_{21}(\mu_2-\mu_1)].$$

It is obvious that $\nabla \neq 0$. Thus, the problem is solved completely.

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