ON SOME GEOMETRICAL PROPERTIES OF MOVING GENERALIZED MÖBIOUS LISTING'S BODIES

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Abstract

The accurate estimation of physical characteristics (such as volume, surface area, length, or other specific parameters) relevant to human organs is of fundamental importance in medicine. The aim of this article is, in this respect, to provide a general methodology for the evaluation, as a function of time, of the volume and center of gravity featured by moving generalized Möbius listing's bodies used to describe different human organs.

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1 Introduction

Recently, the mathematical modeling of human organs is attracting a great deal of interest in modern sciences. As an example, the imaging of heart is essential to collect kinematic and anatomical information in the form of structural boundaries which are afterwards used in suitable computational fluid dynamics models for the evaluation of the blood motion flow within the organ. This in turn is important to analyze heart and valve biodynamics and the mechanisms ensuring their proper functioning. Therefore, it's easy to figure out that the accurate volumetric description of human organs is beneficial for aiding the assessment of the relevant functional behavior and early detection of possible pathologies. In this context, organs can be conveniently modeled in terms of generalized twisted and rotated bodies, whose analytical or algorithmic representations are thoroughly discussed in [6], [7], [9]. In this study, emphasis is put on the evaluation of the geometrical characteristics of these bodies and, in particular, the relevant volume and center of gravity as functions of time due to arbitrary motion. Here it is also worth noting that the displacement of the mentioned bodies can be described analytically in terms of elementary permutations [8], [9].

2 Generalized Twisted and Rotated Bodies

The analytical representation of a generalized twisted and rotated body is given by the following parametric equations [9]:

$$\begin{cases} x(\tau,\psi,\theta,t) = T_1(t) + [R(\theta,t) + p(\tau,\psi,\theta,t)\cos(\psi + n(\theta) + g(t))] \\ \times\cos(\theta + M(t))\,, \\ y(\tau,\psi,\theta,t) = T_1(t) + [R(\theta,t) + p(\tau,\psi,\theta,t)\cos(\psi + n(\theta) + g(t))] \\ \times\sin(\theta + M(t))\,, \\ z(\tau,\psi,\theta,t) = T_3(t) + K(\theta,t) + p(\tau,\psi,\theta,t)\sin(\psi + n(\theta) + g(t))\,, \end{cases} \tag{1}$$

where x, y, z denote, as usual, the Cartesian coordinates, and τ, ψ, θ are spatial parameters satisfying the following conditions:

1. $\tau \in [\tau_*, \tau^*]$, with $\tau_* \leq \tau^*$;
2. $\psi \in [0, 2\pi]$;
3. $\theta \in [0, 2\pi h]$, with $h \in \mathbb{R}$.

In (1), t denotes time, and M, R, p, g, n, K, T_i $(i = 1, 2, 3)$ are sufficiently smooth functions which can be selected, at some extent, in an arbitrary way. The physical meaning and frames of arbitrariness of such functions have been extensively discussed in previously published papers [8], [9]. However, for the sake of clarity, a short description of the relevant features is provided hereafter.

- The function $R(\theta, t)$ defines the basic line of the body, so that:

  – If it is a periodic function of θ, the body is closed or, equivalently, its boundary is a closed surface (see Fig. 1c, f). In the opposite case, the considered body is not closed and its boundary is an open surface (see Fig. 1a, b, d, e and Fig. 2e. break in medical terms);

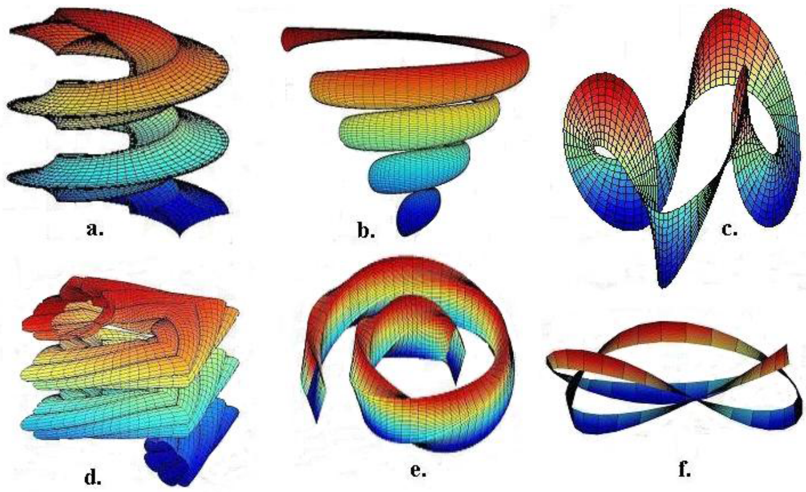

Fig. 1

– If it is a periodic function of t, the body is subject to periodic contractions and expansions (pulsing motion).

• The function $p(\tau, \psi, \theta, t)$ defines the shape of the radial cross section of the body, and:

– If it is a periodic function of t, the cross section of the body is subject to a pulsing motion ("Peristalsis" in medical terms);

– If it is a periodic function of ψ, the cross section of the body is a closed plane figure. In the opposite case, the cross section is open and, in medical terms, one is observing a "rupture" or "scar formation";

– This function defines the size, namely maximal and minimal diameters, of the cross section. In practical scenarios, it is not zero (unless we are considering a "sealed" organ) and, on the other hand, should be within a reference range (conversely, we might be observing an "aneurysm" or a "burst");

– The dependence of this function on the variable θ is to be in accordance with the behavior of the function $R(\theta, t)$. In particular, if $R(\theta, t)$ is a periodic function of θ, then $p(\tau, \psi, \theta, t)$ has to be periodic with the same period, otherwise the surface of the body is open (see Fig. 2a and c).

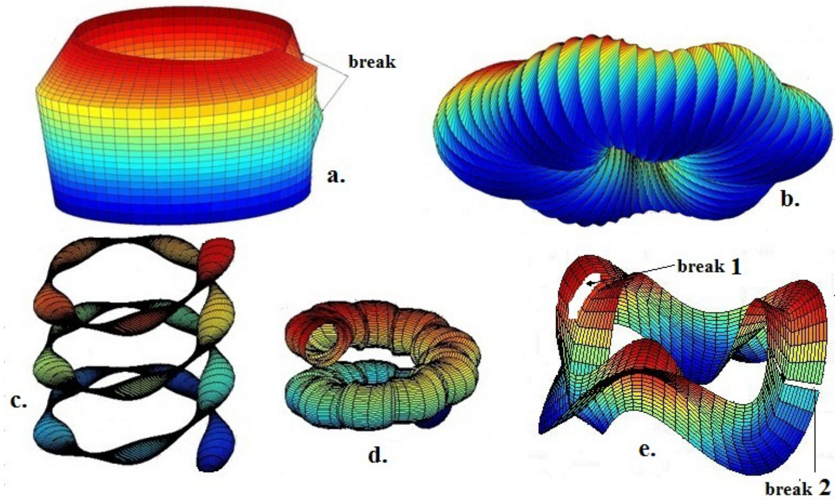

Fig. 2

• The function $g(t)$ defines the twisting characteristics of the radial cross section of the body.

• The function $M(t)$ affects the rotation or swinging of the body around the coordinate Oz axis or, in medical terms, the "rhythm" "or velocity of twisting".

• The function $K(\theta, t)$ affects the vertical stretch of the body, so that:

– If it is identically equal to zero, then then basic line of the body lies in a plane (see Fig. 1e and Fig. 2a, b);

– If it is a $2\pi h$-periodic function, then the considered body features is characterized by a basic line which evolves in the three-dimensional space (see Fig. 1c, f and Fig. 2e);

– If it is a constant, then the basic line is a helix line with fixed pitch (see Fig. 1a, d) or a spiral line on a conic surface (see Fig. 1b and Fig. 2c, d);

– If it is a periodic function of t, the considered body shrinks and expands rhythmically along the coordinate Oz axis (determining the "pulsing" or "contraction" rhythm of the organ).

3 Volume of Generalized Twisted and Rotated Bodies

In the general case, the analytical evaluation of the geometrical parameters relevant to complex-shaped organs as functions of time due to arbitrary motion in the three-dimensional space is not a trivial problem. For the specific class of bodies analyzed in this study, the capacity, or volume, may be calculated according to the classical formula:

$$V(t) = \iiint dxdydz = \int_0^{2\pi h} \int_0^{2\pi} \int_{\tau_*}^{\tau^*} \frac{\partial(x,y,z)}{\partial(\tau,\psi,\theta)} d\tau d\psi d\theta \,, \tag{2}$$

involving the Jacobian determinant:

$$\frac{\partial(x,y,z)}{\partial(\tau,\psi,\theta)} = J(\tau,\psi,\theta) =$$
$$= \begin{vmatrix} p_\tau c(\Psi)C(\Theta) & -(R+pc(\Psi))S(\Theta) + (R_\theta + p_\theta c(\Psi) - pn_\theta s(\Psi))C(\Theta) & -ps(\Psi)C(\Theta) \\ p_\tau c(\Psi)S(\Theta) & (R+pc(\Psi))C(\Theta) + (R_\theta + p_\theta c(\Psi) - pn_\theta s(\Psi))S(\Theta) & -ps(\Psi)S(\Theta) \\ p_\tau s(\Psi) & K_\theta + p_\theta s(\Psi) + pn_\theta c(\Psi) & pc(\Psi) \end{vmatrix}, \tag{3}$$

where $p_\tau \equiv \frac{\partial p}{\partial \tau}$, $c(\Psi) \equiv \cos(\psi + n(\theta) + g(t))$, $s(\Psi) \equiv \sin(\psi + n(\theta) + g(t))$, $C(\Theta) \equiv \cos(\theta + M(t))$, and $S(\Theta) \equiv \sin(\theta + M(t))$. However, after some mathematical manipulations, it is not difficult to derive the expression of the volume of a generalized twisted and rotated body as follows:

$$\begin{aligned} V(t) = & \int_0^{2\pi h} \int_0^{2\pi} \int_{\tau_*}^{\tau^*} (R(\theta,t) + p(\tau,\psi,\theta,t)\cos(\psi + n(\theta) + g(t))) \\ & \times p(\tau,\psi,\theta,t) \frac{\partial p(\tau,\psi,\theta,t)}{\partial \tau} d\tau d\psi d\theta \,, \end{aligned} \tag{4}$$

or, equivalently:

$$\begin{aligned} V(t) = & \int_0^{2\pi h} \int_0^{2\pi} \int_{\tau_*}^{\tau^*} (R(\theta,t) + p(\tau,\psi,\theta,t)\cos(\psi + n(\theta) + g(t))) \\ & \times p(\tau,\psi,\theta,t) dp d\psi d\theta \,, \end{aligned} \tag{5}$$

Remark 1. *If the function $p(\tau,\psi,\theta,t)$ does not depend on ψ, the radial cross section of the body under analysis is a circle whose diameter is, in general, the function of θ (see Fig. 2b, c, d). In this case, since $\int_0^{2\pi} \cos(\psi + n(\theta) + g(t)) d\psi = 0$, the expression (5) assumes the classical form* [7]*:*

$$V(t) = \pi \int_0^{2\pi} R(\theta,t) \left[p^2(\tau^*,\theta,t) - p^2(\tau_*,\theta,t) \right] d\theta \,. \tag{6}$$

4 Center of Gravity of Generalized Twisted and Rotated Bodies

Of great interest in practical applications is the center of gravity, whose Cartesian coordinates can be readily evaluated, under the assumption $T_i(t) = 0$ $(i = 1, 2, 3)$, as:

$$x_c(t) = \iiint x dx dy dz, \;\; y_c(t) = \iiint y dx dy dz, \;\; z_c(t) = \iiint z dx dy dz \,. \tag{7}$$

Upon combining (1) with (7), one easily obtains:

$$x_c(t) = \int_0^{2\pi h} \int_0^{2\pi} \int_{\tau_*}^{\tau^*} (R(\theta, t) + p(\tau, \psi, \theta, t) \cos(\psi + n(\theta) + g(t)))^2 \cdot \\ \cdot p(\tau, \psi, \theta, t) \cos(\theta + M(t)) dp d\psi d\theta \,, \tag{8}$$

$$y_c(t) = \int_0^{2\pi h} \int_0^{2\pi} \int_{\tau_*}^{\tau^*} (R(\theta, t) + p(\tau, \psi, \theta, t) \cos(\psi + n(\theta) + g(t)))^2 \cdot \\ \cdot p(\tau, \psi, \theta, t) \sin(\theta + M(t)) dp d\psi d\theta \,, \tag{9}$$

$$z_c(t) = \int_0^{2\pi h} \int_0^{2\pi} \int_{\tau_*}^{\tau^*} (R(\theta, t) + p(\tau, \psi, \theta, t) \cos(\psi + n(\theta) + g(t))) \cdot \\ \cdot (K(\theta, t) + p(\tau, \psi, \theta, t) \sin(\psi + n(\theta) + g(t)))\, p(\tau, \psi, \theta, t) dp d\psi d\theta \,. \tag{10}$$

By using equations (8)-(10), once can rigorously determine the location of the center of gravity in space and, in this way, infer the type of motion made by the body under analysis.

Remark 2. *If the functions R, p, n do not depend on θ, the basic line of the body is a circle (with possibly time-dependent diameter), whereas the radial cross section changes only in time with constant twisting rate. In this case, upon noticing that $\int_0^{2\pi} \cos(\theta + M(t)) d\theta = 0$, it is easy to figure out that the coordinates x_c and y_c are identically equal to zero, so that center of gravity can move only along the Oz axis.*

Remark 3. *In case the assumptions in **Remark 1** and **Remark 2** hold true and the function K is either identically equal to zero (the basic line of the body is a circle lying on the Oxy plane) or 2π-periodic with respect to the azimuthal angle θ, the coordinate z_c is, also, equal to zero, so that the center of gravity of the body is fixed.*

The determination of the surface area of the bodies considered in the presented study is not trivial, since entails evaluating integral expressions, not reported here for the sake of brevity, which involve the first quadratic form of the mapping (1) from the τ, ψ, θ coordinates to the Cartesian ones x, y, z, and cannot be reduced to simple formulas.

5 Laplacian Operator on Generalized Twisted and Rotated Bodies

Many problems of mathematical physics and electromagnetics are related to the Laplacian operator. Among them, it is worth mentioning those relevant to the Laplace and Helmholtz, as well as heat and wave equations [1]-[6]. In order to address these differential problems in generalized twisted and rotated bodies, a suitable analytical expression of the Laplacian is helpful. In this contribution, this task is accomplished by introducing two different local coordinate system which directly follow from the representation in (1).

First Approach. A system of local coordinates conformal to the "twisted torus" having radius R and twisting parameter μ is introduced, in accordance with (1), as follows:

$$\begin{cases} x = (R + \tau\cos(\psi + \mu\theta))\cos(\theta))\,, \\ y = (R + \tau\cos(\psi + \mu\theta))\sin(\theta))\,, \\ z = \tau\sin(\psi + \mu\theta)\,, \end{cases} \tag{11}$$

so that the inverse mapping is readily found to be:

$$\begin{cases} \theta = \arctan\frac{y}{x}\,, \\ \tau = \sqrt{z^2 + \left(\sqrt{x^2+y^2} - R^2\right)^2}\,, \\ \psi = \arctan\frac{z}{\sqrt{x^2+y^2} - R} - \mu\arctan\frac{y}{x}\,. \end{cases} \tag{12}$$

The coefficients of the Laplacian operator are listed in the second column of Table 13, in comparison with the corresponding coefficients in classical toroidal, as well as spherical coordinates.

Terms	*Twisted Torus*	*Classical Torus*	*Sphere*
$u_{\tau\tau}$	1	1	1
$u_{\theta\theta}$	$(R+\tau\cos(\psi+\mu\theta))^{-2}$	$(R+\tau\cos\psi)^{-2}$	$(\tau\cos\psi)^{-2}$
$u_{\psi\psi}$	$\tau^{-2}+\mu\tau^{-4}(R+\tau\cos(\psi+\mu\theta))^{-2}$	τ^{-2}	τ^{-2}
$u_{\tau\theta}$	0	0	0
$u_{\tau\psi}$	0	0	0
$u_{\theta\psi}$	$-\mu(R+\tau\cos(\psi+\mu\theta))^{-2}$	0	0
u_{τ}	$\frac{1}{\tau}\left(2-\frac{R}{R+\tau\cos(\psi+\mu\theta)}\right)$	$\frac{1}{\tau}\left(2-\frac{R}{R+\tau\cos\psi}\right)$	$\frac{2}{\tau}$
u_{θ}	0	0	0
u_{ψ}	$\frac{-\sin(\psi+\mu\theta)}{\tau(R+\tau\cos(\psi+\mu\theta))}$	$\frac{-\sin\psi}{\tau(R+\tau\cos\psi)}$	$\frac{-\sin\psi}{\tau^2\cos\psi}$

Table. 1

Remark 4.

1. In the system of local twisted toroidal coordinates, the coefficients of the Laplace equation depend on each of the coordinates τ, ψ, θ. This prevents the use of the eigenfunction method and separation of variables (with respect to τ, ψ, θ) for the solution of differential problems in the considered class of bodies.

2. The coefficients of the Laplacian operator strongly depend on the twisting parameter μ. Wherein $\mu = 0$, the mentioned coefficients coincide with those obtained in the classical toroidal coordinate system.

3. Wherein $R = 0$, the oefficients of the Laplacian operator coincide with those obtained in the classical spherical coordinate system.

Remark 5. *The following identities hold true:*

$$\begin{aligned} &P(x,y) \equiv x\cos\left(\arctan\tfrac{y}{x}\right) + y\sin\left(\arctan\tfrac{y}{x}\right) \equiv \sqrt{x^2+y^2} \\ &\equiv R + \tau\cos(\psi+\mu\theta)\,, \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\partial P(x,y)}{\partial x} &= \frac{x}{x^2+y^2}P(x,y) \equiv \frac{x}{P(x,y)}, \\ \frac{\partial P(x,y)}{\partial y} &= \frac{y}{x^2+y^2}P(x,y) \equiv \frac{y}{P(x,y)}, \\ \frac{\partial^2 P(x,y)}{\partial x^2} &= \frac{y^2}{(x^2+y^2)^2}P(x,y) \equiv \frac{y^2}{P^3(x,y)}, \\ \frac{\partial^2 P(x,y)}{\partial y^2} &= \frac{x^2}{(x^2+y^2)^2}P(x,y) \equiv \frac{x^2}{P^3(x,y)}, \\ \frac{\partial^2 P(x,y)}{\partial x \partial y} &= -n\frac{xy}{(x^2+y^2)^2}P(x,y) \equiv -\frac{xy}{P^3(x,y)}. \end{aligned} \tag{14}$$

Second Approach. A system of local coordinates conformal to the"twisted torus-like body" having radial cross section with rectangular shape ($0 < \tau < \tau^*$ and $0 < \varrho < \varrho^*$):

$$\begin{cases} x = (R + \tau\cos(\mu\theta) - \varrho\sin(\mu\theta))\cos(\theta)\,, \\ y = (R + \tau\cos(\mu\theta) - \varrho\sin(\mu\theta))\sin(\theta)\,, \\ z = \tau\sin(\mu\theta) + \varrho\cos(\mu\theta)\,. \end{cases} \tag{15}$$

In this case, the inverse transformation is described by the following equations:

$$\begin{cases} \theta = \arctan\frac{y}{x}\,, \\ \tau = z\sin\left(\mu\arctan\frac{y}{x}\right) + \left(\sqrt{x^2+y^2} - R\right)\cos\left(\mu\arctan\frac{y}{x}\right)\,, \\ \varrho = z\cos\left(\mu\arctan\frac{y}{x}\right) - \left(\sqrt{x^2+y^2} - R\right)\sin\left(\mu\arctan\frac{y}{x}\right)\,, \end{cases} \tag{16}$$

so that the expression of the Laplacian operator is found to be:

$$\begin{aligned}\Delta u \equiv & \left[1+\frac{\mu^2\varrho^2}{R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta)}\right]\frac{\partial^2 u}{\partial\tau^2}\\ & +\left[1+\frac{\mu^2\tau^2}{R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta)}\right]\frac{\partial^2 u}{\partial\varrho^2}\\ & +\frac{1}{(R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta))^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\mu\varrho}{(R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta))^2}\frac{\partial^2 u}{\partial\tau\partial\theta}\\ & -\frac{\mu\varrho}{(R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta))^2}\frac{\partial^2 u}{\partial\varrho\partial\theta}-\frac{\mu\tau\varrho}{(R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta))^2}\frac{\partial^2 u}{\partial\tau\partial\varrho}\\ & +\left[\frac{-\mu^2\tau}{(R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta))^2}+\frac{\cos(\mu\theta)}{R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta)}\right]\frac{\partial u}{\partial\tau}\\ & +\left[\frac{-\mu^2\varrho}{(R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta))^2}-\frac{\sin(\mu\theta)}{R+\tau\cos(\mu\theta)-\varrho\sin(\mu\theta)}\right]\frac{\partial u}{\partial\varrho}.\end{aligned} \tag{17}$$

Remark 6.

1. In the system of local twisted toroidal-like coordinates, the coefficients of the Laplace equation depend on each of the coordinates τ, ϱ, θ. This prevents the use of the eigenfunction method and separation of variables (with respect to τ, ϱ, θ) for the solution of differential problems in the considered class of bodies.

2. The coefficients of the Laplacian operator strongly depend on the twisting parameter μ. Wherein $\mu = 0$ (the radial cross section of the body doesn't twist), the mentioned coefficients coincide with those obtained in the classical cylindrical

6 About Some Geometrical Properties of Some Subclasses of Surfaces GTR_2^n

In this section,attention is put on a particular case of "non-regular" generalized twisting and rotated surfaces GTR_2^n described by the following parametric equations:

$$\begin{cases} x(\tau,\psi,\theta) = (R+\tau\cos(\psi+\mu n(\theta)))\cos(\theta)\,, \\ y(\tau,\psi,\theta) = (R+\tau\cos(\psi+\mu n(\theta)))\sin(\theta)\,, \\ z(\tau,\psi,\theta) = K(\theta)+\tau\sin(\psi+\mu n(\theta))\ , \end{cases} \tag{18}$$

where R is the radius of orthogonal projection of the basic line of the surface, $n(\theta)$ denotes an arbitrary twisting function with coefficient $\mu = k/2$ ($k \in \mathbf{Z}$), and $K(\theta)$ ia a sufficiently smooth function which affects the vertical stretch of the surface. In (18) the variables τ, ψ, θ are assumed to satisfy the same conditions listed in the section 2 for the representation (1).

After some mathematical manipulations, it can be shown that the tangential vectors of the general surface belonging to the considered class are given by:

$$\vec{\mathbf{r}}_\tau = \left\{ \begin{array}{l} \cos(\psi + \mu n(\theta)) \cos(\theta) \\ \cos(\psi + \mu n(\theta)) \sin(\theta) \\ \sin(\psi + \mu n(\theta)) \end{array} \right\}, \tag{19}$$

and:

$$\vec{\mathbf{r}}_\theta = \left\{ \begin{array}{l} -(R + \tau\cos(\psi + \mu n(\theta))) \sin(\theta) - \tau\mu n\prime(\theta) \sin(\psi + \mu n(\theta)) \cos(\theta) \\ (R + \tau\cos(\psi + \mu n(\theta))) \cos(\theta) - \tau\mu n\prime(\theta) \sin(\psi + \mu n(\theta)) \sin(\theta) \\ \tau\mu n\prime(\theta) \cos(\psi + \mu n(\theta)) + K'(\theta) \end{array} \right\}, \tag{20}$$

respectively, so that the expression of the relevant scalar product can be easily derived as:

$$(\vec{\mathbf{r}}_\tau, \vec{\mathbf{r}}_\theta) = K'(\theta) \sin(\psi + \mu n(\theta)) . \tag{21}$$

Remark 7. *If $K'(\theta) \equiv 0$, then for any integer number k the tangential vectors $\vec{\mathbf{r}}_\tau$, $\vec{\mathbf{r}}_\theta$ are perpendicular to each other, meaning that the local system of coordinates (τ, θ) is orthogonal:*

a) If $K(\theta) \equiv 0$, we have a generalized Möbius-Listing's surface GML_2^n with cicular basic line;

b) If $K(\theta) \equiv const.$, we have a helicoidal surface with constant vertical stretching.

From equations (19), it also follows that:

$$\frac{\partial(x, y)}{\partial(\tau, \theta)} = (R + \tau\cos(\psi + \mu n(\theta))) \cos(\psi + \mu n(\theta)) , \tag{22}$$

$$\begin{aligned} \frac{\partial(z, x)}{\partial(\tau, \theta)} = &-(R + \tau\cos(\psi + \mu n(\theta))) \sin(\psi + \mu n(\theta)) \sin(\theta) \\ &- \big(\tau\mu n'(\theta) - K'(\theta) \cos(\psi + \mu n(\theta))\big) \cos(\theta) , \end{aligned} \tag{23}$$

$$\begin{aligned} \frac{\partial(y, z)}{\partial(\tau, \theta)} = &-(R + \tau\cos(\psi + \mu n(\theta))) \sin(\psi + \mu n(\theta)) \cos(\theta) \\ &+ \big(\tau\mu n'(\theta) + K'(\theta) \cos(\psi + \mu n(\theta))\big) \sin(\theta) , \end{aligned} \tag{24}$$

and correspondingly the module of the vector product of these two vectors is found to be:

$$|\vec{\mathbf{r}}_\tau \times \vec{\mathbf{r}}_\theta| = \sqrt{(R + \tau\cos(\psi + \mu n(\theta)))^2 + (\tau\mu n'(\theta)/2 + K'(\theta) \cos(\psi + \mu n(\theta)))^2} . \tag{25}$$

Using equations (22)-(25), it is not difficult to show that:

Remark 8. *The following properties of the surfaces (18) hold true:*

a) If $K(\theta) \neq 0$ and $K(\theta)$ is not a 2π-periodic function, then the corresponding helicoidal surface is two-sided for any integer index n;

b) If $K(\theta) \equiv 0$ or $Q(\theta)$ is a 2π-periodic function, then the corresponding generalized Möbius-Listing's surface GML_2^n is two-sided (the unit normal vector is a 2π-periodic function);

c) If $K(\theta) \equiv 0$ or $Q(\theta)$ is a 2π-periodic function, then the unit normal vector is a 4π-periodic function and the corresponding generalized Möbius-Listing's surface GML_2^n is one-sided;

The first fundamental form of the general surface belonging to the considered class is described by the equations:

$$E(\tau,\theta) = 1\,, \tag{26}$$

$$F(\tau,\theta) = K' \sin\left(\psi + \mu n(\theta)\right)\,, \tag{27}$$

$$\begin{aligned} G(\tau,\theta) = {} & (R + \tau\cos(\psi + \mu n(\theta)))^2 + K'^2(\theta)\sin^2(\psi + \mu n(\theta)) \\ & + \left(\tau\mu n'(\theta) + K'^2(\theta)\cos(\psi + \mu n(\theta))\right)^2\,, \end{aligned} \tag{28}$$

so that:

$$EG - F^2 = (R + \tau\cos(\psi + \mu n(\theta)))^2 + \left(\tau\mu n'(\theta) + K'^2(\theta)\cos(\psi + \mu n(\theta))\right)^2\,. \tag{29}$$

Remark 9. *Each point of a GML_2^n surface (18) is regular.*

The second fundamental form of this class of surfaces is given by:

$$L(\tau,\theta) = 0\,, \tag{30}$$

$$M(\tau,\theta) = \frac{2\mu R n'(\theta) - K'(\theta)\cos(\psi + \mu n(\theta))}{\sqrt{EG - F^2}}\,, \tag{31}$$

$$\begin{aligned} N(\tau,\theta) = {} & \frac{1}{\sqrt{EG - F^2}}\Big\{(R + \tau\cos(\psi + \mu n(\theta)))^2 \sin(\psi + \mu n(\theta)) \\ & + (R + \tau\cos(\psi + \mu n(\theta)))\left(\tau\mu^2 n''(\theta)) + K''(\theta)\cos(\psi + \mu n(\theta))\right) \\ & + \tau n'(\theta)\left(\tau n'(\theta) + K'(\theta)\cos(\psi + \mu n(\theta))\sin(\psi + \mu n(\theta))\right)\Big\}\,. \end{aligned} \tag{32}$$

Furthermore, we may rewrite the mean and Gaussian curvatures of this class of surfaces

$$G(\tau,\theta) = \frac{-\left[2\mu R n'(\theta) - K'(\theta)\cos\left(\psi + \frac{ng(\theta)}{2}\right)\right]^2}{\left[EG - F^2\right]^2}\,. \tag{33}$$

From equation (33) it is clear that:

Remark 10. *The following properties hold true:*

a) Each point of a GTR_2^n surface (18) is parabolic or hyperbolic (saddle) point;

b) If $K(\theta) \equiv const.$, then the considered surface features only saddle points.

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