SOLUTION OF A TWO-DIMENSIONAL PROBLEM OF THE THEORY OF STATIONARY LIQUID FILTRATION THROUGH AN EARTH DAM WITH THE BROKEN BACKSLOPE

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Abstract

The paper proposes a theoretical solution scheme for a two-dimensional problem of the theory of stationary liquid filtration through an earth dam. The dam foundation is water-proof. The dam backslope is the broken line consisting of two segments of the straight line which forms with the dam foundation an angle $\pi/2$, whereas the tail-water level is equal to zero.

Key words and phrases: Filtration, analytic function, generalized analytic function, conformal mapping, differential equation.

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1 Introduction

In this paper we obtain an analytic solution of a mixed two-dimensional problem of the theory of stationary filtration through a plain earth dam with a backslope and a partially unknown boundary. The tail-water level is assumed to be equal to zero. The dam foundation is assumed to be water-proof, and the porous medium to be isotropic, homogeneous and non-deformable. Liquid motion in the porous medium obeys the Darcy law. The boundary $l(z)$ of the domain $S(z)$ occupied by the moving fluid consists of an unknown depression curve and the known straight line segments. The fluid motion scheme is shown in Fig.1.

In the domain $S(z)$ with boundary $l(z)$ we define the reduced complex potential (the potential divided by the constant filtration coefficient $\omega(z) = \varphi(x, y) + i\psi(x, y)$ where the potential velocity $\varphi(x, y)$ and the flow function $\psi(x, y)$ satisfy the Cauchy-Riemann conditions and the boundary conditions [1-7, 15-22]. We derive analytic formulas for calculation of the geometric and mechanical characteristics (parameters). The parametric equations are derived for calculating the unknown part of the boundary and the depression curve. In solving the problem, we widely use the theory of differential equations belonging to the Fuchs class and also non-linear Schwartz differential equations.

2 Statement of the Problem

In this paper we give a Frolov type theoretical scheme for solution of the two-dimensional problem of the theory of stationary filtration through an earth dam, the backslope of which is a broken line formed by two segments of the straight line which forms with the dam foundation an angle $\pi/2$, whereas the tail-water level is zero [1–9].

The plane of incompressible fluid motion is combined with the plane of a complex variable $z = x + iy$. The fluid motion obeys the Darcy law. The porous medium is isotropic, homogeneous and non-deformable. The boundary $l(z)$ of the domain $S(z)$ occupied by the moving fluid consists of an unknown depression curve, and the known straight line segments. The fluid motion scheme is shown in Fig.1.

In the domain $S(z)$ with boundary $l(z)$ we seek for a complex potential (a potential divided by the constant filtration coefficient) $\omega(z) = \varphi(x, y) + i\psi(x, y)$, where the velocity potential $\varphi(x, y)$ and the flow function $\psi(x, y)$ satisfy the Cauchy-Riemann equations and the boundary conditions [15-22]

$$a_{k1}\varphi(x, y) + a_{k2}\psi(x, y) + a_{k3}x + a_{k4}y = f_k,\ k = 1, 2,\ (x, y) \in l(z), \quad (1.1)$$

where $a_{ki},\ f_k,\ k = 1, 2;\ i = \overline{1, 4}$, are the known piecewise-constant real functions.

We denote by $S(\omega)$ and $S(w)$ respectively the domains of the complex potential and the complex velocity, and by $l(\omega)$, $l(w)$ their respective boundaries.

The angular points of the boundaries $l(z)$, $l(\omega)$ and $l(w)$ which may occur on one of them at least when they are bypassed in the positive positive direction are denoted by A_k, $k = \overline{1, 6}$.

The boundary conditions for the considered problem are as follows:A_1A_2 : $\varphi(x, y) + y = 0,\ y = -\tan(\pi\beta)(x - L)$ along the seepage path, where H is the head-water depth, L is the dam foundation length, $\pi\beta$ is the internal angle at the point A_2; $A_2A_3A_4 : \varphi(x, y) + y = 0,\ \psi(x, y) = Q$ along the depression curve; $A_4A_5 : \varphi(x, y) = -H,\ y = \tan(\pi\alpha)(x + L_1)$, where $L_1 = A_5A_6$, along the head-water level $A_5A_6 : \varphi(x, y) = -H,\ x = 0$, the angle is equal to $\pi(1/2 + \alpha)$ at the point A_5 and to $\pi/2$ at the point A_6; $A_6A_1 : \psi(x, y) = 0,\ y = 0$ along the dam foundation (Fig.1). It is proved in [1–21] that the domain $S(w)$ with boundary $l(w)$ is the circular pentagon (Fig. 2).

A half-plane $I_m(\zeta) \geq 0$ of the plane $\zeta = t + i\tau$ is conformally mapped onto the domains $S(z)$, $S(\omega)$ and $S(w)$, whereas $z(\zeta)$, $\omega(\zeta)$, $w(\zeta) = \omega'(\zeta)/z'(\zeta)$ denote the respective conformally mapping functions. To the angular points A_k, $k = \overline{1,6}$ along the t-axis there correspond the points $t = e_k$, $k = \overline{1,6}$; also, $-\infty < e_1 < e_2 < \cdots < e_5 < e_6 < +\infty$, where $t = \infty$ is mapped into the point $A_7(e_7)$, $e_7 = \infty$.

When $\zeta \to t$, $\zeta \in I_m(\zeta) > 0$, the boundary values of the functions $z(\zeta)$, $\omega(\zeta)$ and $w(\zeta)$, are denoted as follows: $z(t) = x(t) + iy(t)$, $\omega(t) = \varphi(t) + i\psi(t)$, $w(t) = u(t) - iv(t)$, where $u(t)$, $v(t)$ are the velocity vector components; the complex conjugate functions to the functions $z(t)$, $\omega(t)$ and $w(t)$ are respectively denoted by $\overline{z(t)}$, $\overline{\omega(t)}$ and $\overline{w(t)}$.

Let us introduce the vectors $\Phi(t) = [\omega(t), z(t)]$, $\overline{\Phi(t)} = [\overline{\omega(t)}, \overline{z(t)}]$, $\Phi'(t) = [\omega'(t), z'(t)]$, $\overline{\Phi'(t)} = [\overline{\omega'(t)}, \overline{z'(t)}]$, $f(t) = [f_1(t), f_2(t)]$. Then the boundary conditions (1.1) take the form:

$$\Phi(t) = g(t)\,\overline{\Phi(t)} + f(t), \quad -\infty < t < +\infty, \tag{1.2}$$

$$\Phi'(t) = g(t)\,\overline{\Phi'(t)}, \quad -\infty < t < +\infty, \tag{1.3}$$

where

$$g(t) = G^{-1}(f)\,\overline{G(f)}, \quad \overline{g(t)} = \overline{G^{-1}(t)}\,G(t), \tag{1.4}$$

$$g_\infty(t) = E, \quad f_{-\infty}(t) = [0,0], \quad -\infty < t < e_1,$$

$$g_1(t) = \begin{pmatrix} -1, & 2\sin(\pi\beta)\exp(-i\pi\beta) \\ 0, & \exp(-i2\pi\beta) \end{pmatrix}, \quad e_1 < t < e_2, \tag{1.5}$$

$$f_1(t) = 2L\sin(\pi\beta)\exp(-i\pi\beta)[-1, i], \quad e_1 < t < e_2,$$

$$g_2(t) = g_3(t) = \begin{pmatrix} 1, & 0 \\ -2i, & 1 \end{pmatrix}, \ f_2(t) = f_3(t) = 2Q[i,1], \ e_2 < t < e_3 < e_4,$$

$$g_4(t) = \begin{pmatrix} -1, & 0 \\ 0, & \exp(i2\pi\alpha) \end{pmatrix}, \ f_4(t) = 2[-H, ie^{i\pi\alpha}\cos(\pi\alpha)H], \ e_4 < t < e_5,$$

$$g_5(t) = \begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}, \quad f_5(t) = -2H[1.0],$$

where

$$E = \begin{pmatrix} -1 & 0 \\ 0, & -1 \end{pmatrix}.$$

For the points $t = e_i$, $i = \overline{1,6}$, consider the characteristic equation [16–21],

$$\det\left|g_{i+1}^{-1}(e_i + 0)g_i(e_i - 0) - \lambda E\right| = 0, \tag{1.6}$$

with respect to the parameter λ, where E is the unit matrix, $g_i(t)$, $e_i < t < e_{i+1}$, $g_{i+1}^{-1}(e_i + 0), g_i(e_i - 0)$ are the limiting values of the matrices $g_{i+1}^{-1}(t)$,

$g_i(t)$ at the point e_i, to the right and to the left, respectively. The numbers $x_{ki} = (2\pi i)^{-1} \ln \lambda_{ki}$, $k = 1, 2$, $i = \overline{1, 6}$, are uniquely defined using the roots λ_{ki} of equation (1.6)[8-15, 16-19]. At the angular points $A_i[\alpha_{1i}, \alpha_{2i}]$, $i = \overline{1, 6}$, and at the non-angular point $\zeta = \infty$ the characteristic exponents for the functions $\omega'(\zeta)$, $z'(\zeta)$ have the form:

$$\begin{gathered} A_1[-1/2; \beta - 1], \quad A_2[1/2 - \beta; 0], \quad A_3[2; 0], \quad A_4[-1/2; -a], \\ A_5[0; \alpha - 1/2], \quad A_6[+1/2; -1/2], \quad A_7[3; 2]. \end{gathered} \tag{1.7}$$

The point A_6 is a removable singular point. To remove it we introduce a new vector $\Phi_1(t)$ by the formula [8-15, 16-19]

$$\Phi'(t) = \chi_1(t)\Phi_1(t), \quad -\infty < t < +\infty, \tag{1.8}$$

where

$$\chi_1 = \sqrt{(t - e_5)(t - e_6)^{-1}} > 0, \quad t > e_6. \tag{1.9}$$

The boundary condition with respect to $\Phi_1(t)$ takes the form

$$\Phi_1(t) = g^*(t)\overline{\Phi_1(t)}, \quad -\infty < t < +\infty, \tag{1.10}$$

where

$$g^*(t) = |\chi_1(t)|^{-1} g(t)\overline{\chi_1(t)}. \tag{1.11}$$

The exponents at the points A_5 and A_6 take respectively the form $A_5^*[1/2; \alpha - 1]$, $A_6^*[0, 1]$.

Let us enumerate anew the singular points on the contour $l(w)$ and denote them by B_i, $i = \overline{1, 6}$, and denote the corresponding points along the t-axis by a_i, $i = \overline{1, 6}$, $a_6 = \infty$. Denote by α_{ki}, $k = 1, 2$, $i = \overline{1, 6}$, the characteristic numbers corresponding to the points $t = a_i$, $i = \overline{1, 6}$, which satisfy the Fuchs condition. We introduce the notation

$$\begin{gathered} B_1[-1/2; \beta - 1], \quad B_2[1/2 - \beta; 0], \quad B_3[2; 0], \\ B_4[-1/2; -\alpha], \quad B_5[-1/2; \alpha - 1], \quad B_6[3; 2]. \end{gathered} \tag{1.12}$$

Using (1.12) we write an equation of the Fuchs class [1-6, 17-22]

$$u''(\zeta) + P(\zeta)\, u'(\zeta) + g(t)\, u(\zeta) = 0, \tag{1.13}$$

where

$$p(\zeta) = \sum_{i=1}^{5} \beta_i (\zeta - a_i)^{-1}, \quad \beta_i = 1 - \alpha_{1i} - \alpha_{2i}, \tag{1.14}$$

$$g(\zeta) = \sum_{i=1}^{5} \left[\alpha_{1i}\alpha_{2i}(\zeta - a_i)^{-2} + c_i(\zeta - a_i)^{-1}\right]. \tag{1.15}$$

where c_i are the unknown accessory parameters, satisfying the condition

$$M = \sum_{i=1}^{5} c_i = 0. \tag{1.16}$$

Using the linearly independent solutions $u_1(t)$ and $u_2(t)$ of equation (1.13), we construct a general solution

$$w(\zeta) = \left[pu_1(\zeta) + qu_2(\zeta)\right]\left[ru_1(\zeta) + su_2(\zeta)\right]^{-1} \tag{1.17}$$

of the Schwartz equation [15-22]

$$\{w(\zeta), \zeta\} = w'''(\zeta)/w'(\zeta) - 1,5(w''(\zeta)/w'(\zeta))^2 = R(\zeta), \tag{1.18}$$

where

$$R(\zeta) = 2q(\zeta) - p'(\zeta) - 0,5[p(\zeta)]^2$$

$$= \sum_{i=1}^{5} \left\{0,5[1-(\alpha_{1i}-\alpha_{2i})^2](\zeta-a_i)^{-2} + c^*(\zeta-a_i)^{-1}\right\}, \tag{1.19}$$

$$\alpha_{1i} - \alpha_{2i} = \nu_i, \quad i = \overline{1,5}, \quad c_i^* = 2c_i - \beta_i \sum_{i=1}^{5}(a_i - a_k)^{-1}. \tag{1.20}$$

From (1.19) it follows that (1.14) depends on the difference $\alpha_{1i} - \alpha_{2i} = \nu_i$, $i = \overline{1,5}$, where $\pi\nu i$ is the internal angle at the point B_i of the circular polygon. p, q, r and s are the integration constants of (1.18) which satisfy the condition $ps - qr \neq 0$.

Among the points $t = a_i$, $i = \overline{1,5}$, by the Riemann theorem we arbitrarily choose and fix three of them. In our case, the parameters $t = a_k$, $k = \overline{1,5}$, are fixed as follows

$t = a_i$, $i = \overline{1,5}$, $a_1 = -b$, $a_2 = -a$, $a_3 = 0$, $a_4 = a$, $a_5 = b$.

Since the point $\zeta = a_6 = \infty$ is the image of a non-angular point of the boundary $l(w)$, the following conditions must be fulfilled [11, 13]:

$$M_1 \equiv \sum_{k=1}^{5} c_k^* = 0, \tag{1.21}$$

$$M_2 \equiv \sum_{k=1}^{5} \left[a_k c_k^* + 0,5(1-\nu_k^2)\right] = 0, \tag{1.22}$$

$$M_3 \equiv \sum_{k=1}^{5} \left[a_k^2 c_k^* + a_k(1-\nu_k^2)\right] = 0. \tag{1.23}$$

Condition (1.21) implies (1.16) and, conversely, condition (1.16) implies (1.21).

Below we will obtain conditions (1.21)–(1.23) in a different manner [16–21]. From conditions (1.22)–(1.23) we define three parameters c_i, $i = \overline{1,3}$, and therefore $R(\zeta)$ will depend on the four unknown parameters a, b, c_4, c_5.

Equation (1.13) near the points $t = a_i$, $i = \overline{1,5}$, can be rewritten as

$$(t-a_i)^2 u''(t) + (t-a_i) p_i(t)\, u'(t) + q_i(t)\, u(t) = 0, \tag{1.24}$$

where

$$\begin{gathered} p_i(t) = p_{0i} + \sum_{i=1}^{\infty} p_{ni}(t-a_i)^n, \\ p_{ni} = (-1)^n \sum_{k=1,k\neq i}^{5} \beta_k (a_i - a_k)^{-1}, \end{gathered} \tag{1.25}$$

$$q_i = \alpha_{1i}\alpha_{2i} + c_i(t-a_i) + \sum_{n=2}^{\infty} q_{ni}(t-a_i)^n, \quad q_{0i} = \alpha_{1i}\alpha_{2i}, \quad q_{1i} = c_i,$$
$$i = \overline{1,5}, \quad n = 0,1,$$

$$q_{ni} = (-1)^{n-2} \sum_{k=2,k\neq i}^{5} \big[\alpha_{1k}\alpha_{2k}(n-1) + c_k(a_i - a_k)\big](a_i - a_k)^{-n}, \tag{1.26}$$
$$n = 2, 3, \dots \ .$$

Local solutions of (1.24) for the points $t = a_i$, $i = \overline{1,5}$, are sought in the form

$$u_i(t) = (t-a_i)^{\alpha_i}\widetilde{u}_i(t), \quad \widetilde{u}_i(t) = 1 + \sum_{n=1}^{\infty} \gamma_{ni}(t-a_i)^n, \tag{1.27}$$

where γ_{ni}, $n = \overline{1,\infty}$, $i = \overline{1,5}$, are defined by the following recurrent formulas:

$$f_{0i}(\alpha_i) \equiv \alpha_i(\alpha_i - 1) + p_{0i}\alpha_i + q_{0i} = 0, \tag{1.28}$$
$$\gamma_{1i} f_{0i}(\alpha_i + 1) + f_{1i}(\alpha_i) = 0, \tag{1.29}$$
$$\gamma_{2i} f_{0i}(\alpha_i + 2) + \gamma_{1i} f_{1i}(\alpha_i + 1) + f_{2i}(\alpha_i) = 0, \tag{1.30}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where

$$f_n(\alpha_i) = \alpha_i p_{ni} + q_{ni}. \tag{1.31}$$

If the difference $\alpha_{1i} - \alpha_{2i}$, $i = \overline{1,5}$, is not an integer number, then using (1.28)–(1.31) we construct the linearly independent solutions (1.27),

$$u_{ki}(t) = (t - a_i)^{\alpha_{ki}} \widetilde{u}_{ki}(t), \quad \widetilde{u}_{ki}(t) = 1 + \sum_{n=1}^{\infty} \gamma_{ni}^{k} (t - a_i)^n, \tag{1.32}$$
$$k = 1, 2; \quad i = \overline{1,5}.$$

If however $\alpha_{1i} - \alpha_{2i} = n$, $n = 0, 1, 2$, then $u_{1i}(t)$ is constructed by (1.28)–(1.31), whereas $u_{2i}(t)$ is constructed by the Frobenius method [10, 13]. When $\alpha_{1i} - \alpha_{2i} = 0$, we have [16–21]

$$u_{2i}(t) = u_{1i}(t) \ln(t - a_i) + (t - a_i)^{\alpha_{2i}} \sum_{n=1}^{\infty} \gamma_{ni}^{2} (t - a_i)^n. \tag{1.33}$$

When $\alpha_{1i} - \alpha_{2i} = n$, $n = 1, 2$, to construct $u_{2i}(t)$ we have to differentiate the equality

$$u_{2i}(t) = (t - a_i)^{\alpha_i} \Big[\alpha_i - \alpha_{2i} + \sum_{n=1}^{\infty} \gamma_{ni}(\alpha_i)(t - a_i)^n \Big], \tag{1.34}$$

with respect to α_i and then $\alpha_i \to \alpha_{2i}$. As a result we obtain

$$u_{2i}(t) = (t - a_i)^{\alpha_{2i}} \Big[\sum_{n=1}^{\infty} \lim_{\alpha_i \to \alpha_{2i}} \gamma_n(\alpha_i)(t - a_i)^n \Big] \ln(t - a_i) +$$
$$+ (t - a_i)^{\alpha_{2i}} \Big\{ 1 + \sum_{n=1}^{\infty} \Big[\frac{d\gamma_n(\alpha_i)}{d\alpha_i} \Big]_{\alpha_i = \alpha_{2i}} (t - a_i)^n \Big\}. \tag{1.35}$$

P. J. Polubarinova-Kochina proved that the solution $u_2(t)$ does not contain a logarithmic term all along the cut, and she obtained an algebraic equation relating the parameters a_i, c_i, $i = \overline{1,5}$. For the unique construction of $u_{2i}(t)$, the following method was proposed in [16–21]. For such points the condition $t = a_i$ is not fulfilled since

$$f_{0i}(\alpha_i + 2) = 0, \quad \alpha_i \to \alpha_{2i}.$$

For equality (1.30) to hold as $\alpha_i \to \alpha_{2i}$, it is necessary and sufficient to require the fulfilment of the condition [16–21]

$$\gamma_{1i} f_1(\alpha_i + 1) + f_2(\alpha_i) = 0, \quad \alpha_{1i} - \alpha_{2i} = 2. \tag{1.36}$$

After simplification (1.36) becomes

$$q_{2i} + q_{1i}^2 + q_{1i}\, p_{1i} = 0. \tag{1.37}$$

For the unique construction of $u_{2i}(t)$, it is sufficient to construct uniquely $\gamma_{2i}^2(\alpha_{2i})$, whereas the remaining $\gamma_{ni}^2(\alpha_i)$, $n = 1, 2, 4, \dots$ are calculated by (1.28)–(1.31). Assume that $\alpha_{1i} \neq \alpha_{2i} + 2$, then (1.30) implies [16–21]

$$\gamma_{2i}(\alpha_i) = -\big[\gamma_{1i}(\alpha_i) f_{1i}(\alpha_i + 1) + f_{2i}(\alpha_i)\big] / f_0(\alpha_i + 2). \tag{1.38}$$

On the right-hand side of (1.38) the numerator and denominator vanish for $\alpha_i = \alpha_{2i} + 2$. After evaluation of the indeterminate form in the right-hand parr of (1.38) for $\alpha_i = \alpha_{2i} + 2$, we uniquely define $\gamma_{2i}^2(\alpha_i)$ by

$$\gamma_{2i}^2 = -0,5\big[p_{1i}(p_{1i} + 2q_{1i}) + p_{2i}\big]. \tag{1.39}$$

Let us proceed to constructing a local solution near the point $t = \infty$. Functions $p(t)$ and $q(T)$ near $t = \infty$ are represented as follows:

$$p(t) = t^{-1} \sum_{n=0}^{\infty} p_{n\infty} t^{-n}, \quad q(t) = t^{-2} \sum_{n=0}^{\infty} q_{n\infty} t^{-n}, \tag{1.40}$$

where

$$\begin{aligned} p_{n\infty} &= \sum_{k=1}^{5} \beta_k a_k^n, \quad \beta_k = 1 - \alpha_{1k} - \alpha_{2k}, \quad p_{0\infty} = 6, \\ q_{n\infty} &= \sum_{k=1}^{m} \big[\alpha_{1k}\alpha_{2k}(n+1) + c_k a_k\big] a_k^n. \end{aligned} \tag{1.41}$$

The solution $u_\infty(t)$ will be constructed in the form [16-21],

$$u_\infty(t) = t^{-\alpha_\infty} + \sum_{n=1}^{\infty} \gamma_{n\infty} t^{-(\alpha_\infty + n)}, \tag{1.42}$$

where $\gamma_{n\infty}$, $n = \overline{1, \infty}$, are defined by the formulas

$$f_{0\infty}(\alpha_\infty) \equiv \alpha_\infty(\alpha_\infty + 1) - p_{0\infty}\alpha_\infty + q_{0\infty} = 0, \tag{1.43}$$

$$\gamma_{1\infty} f_{0\infty}(\alpha_\infty + 1) - p_{1\infty}\alpha_\infty + q_{1\infty} = 0, \tag{1.44}$$

$$\gamma_{2\infty} f_{0\infty}(\alpha_\infty + 2) + \gamma_{1\infty}(\alpha_\infty + 1) + p_{2\infty}(\alpha_\infty) + q_{2\infty} = 0, \tag{1.45}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where

$$f_{k\infty} = q_{k\infty} - (\alpha_\infty + k) p_{k\infty}. \tag{1.46}$$

Since the point $t = \infty$ is the image of a non-angular point, equation (1.43) must have the roots $a_{1\infty} = 3$, $a_{2\infty} = 2$, therefore

$$q_{0\infty} = \sum_{k=1}^{5} \big[\alpha_{1k}\alpha_{2k} + a_k c_k\big] = 6. \tag{1.47}$$

Equation (1.44) does not hold since $\alpha_{1\infty}-\alpha_{2\infty}=1$. Formulas (1.43)–(1.46) allow us to define only one solution $u_{1\infty}(t)$. To define $u_{2\infty}(t)$, it is necessary and sufficient to require the fulfilment of the condition

$$q_{1\infty}-p_{1\infty}\alpha_{2\infty}=0. \tag{1.48}$$

To define $\gamma_{1\infty}^2$, we proceed as follows. Using(1.44),where it is assumed that $\alpha_\infty\neq\alpha_{2\infty}$, to define $\gamma_{1\infty}$, we obtain

$$\gamma_{1\infty}=\big[p_{1\infty}\alpha_\infty-q_{1\infty}\big]/f_{0\infty}(\alpha_\infty+1). \tag{1.49}$$

After evaluating the indeterminate form we obtain

$$\gamma_{1\infty}^2=p_{1\infty}. \tag{1.50}$$

Now all $\gamma_{n\infty}$, $n=\overline{1,\infty}$, are defined by formulas (1.43)–(1.46). Hence we define the linearly independent solutions at the point $t=\infty$

$$u_{k\infty}(t)=t^{-\alpha_{k\infty}}+\sum_{n=1}^{\infty}\gamma_{n\infty}^k\,t^{-\alpha_{k\infty}-n},\quad k=1,2. \tag{1.51}$$

The local solutions $u_{ki}(t)$, $k=1,2$; $i=\overline{1,5}$, contain the multi-valued functions among which we choose one-valued functions as follows:

$$\exp\big[\beta_{ki}(t-a_i)\big]>0,\quad t>a_i,$$
$$\big\{\exp\big[\alpha_{ki}\ln(t-a_i)\big]\big\}^+=\exp[i\pi\alpha_{ki}]\big[\exp[\alpha_{ki}\ln(a_i-t)]\big],\ a_i-t>0,$$
$$\big\{\exp\big[\alpha_{ki}\ln(t-a_i)\big]\big\}^-=\exp[-i\pi\alpha_{ki}]\big[\exp[\alpha_{ki}\ln(a_i-t)]\big],\ a_i-t>0.$$

For equation (1.13), near the singular points $t=a_i$, $i=\overline{1,6}$, and near the ordinary points $t=a_i^*=(a_i+a_{i+1})/2$, $i=\overline{1,4}$, in the sequel we will construct $u_{ki}(t)$, $k=1,2$, $i=\overline{1,6}$; $\sigma_{ki}(t)$, $k=1,2$, $i=\overline{1,4}$.

2 Construction Solutions Equation (1.13)

Let us write equation (1.13) in the form [16–21]

$$\chi'(t)=\chi(t)\,P(t), \tag{2.1}$$

where

$$P(t)=\begin{pmatrix}0, & -q(t)\\ 1, & -p(t)\end{pmatrix},\quad \chi(t)=\begin{pmatrix}u_1(t), & u_1'(t)\\ u_2(t), & u_2'(t)\end{pmatrix}. \tag{2.2}$$

Solution (1.10) will be sought by means of the matrix $T\,\chi(t)$ where $\chi(t)$ is the solution of (2.1). If $\chi(t)$ is the solution of (2.1), then $T\chi(t)$, too, is the solution of (2.1), where T is a constant matrix,

$$T = \begin{pmatrix} p, & q \\ r, & s \end{pmatrix}, \quad \det T \neq 0, \tag{2.3}$$

p, q, r and s are the integration constants of (1.18), (1.7) [1-7, 16-22].

The local fundamental matrices $\Theta_i(t)$, $\sigma_i(t)$, $\Theta_i^*(t)$, $\Theta_i^{\pm}(t)$ are defined as follows:

$$\Theta_i(t) = \begin{pmatrix} u_{1i}(t), & u'_{1i}(t) \\ u_{2i}(t), & u'_{2i}(t) \end{pmatrix}, \; a_i < t < a_{i+1}, \; i = \overline{1, i-1}, \quad t = a_i, \; i = \overline{1.5},$$

$$\Theta_i^*(t) = \begin{pmatrix} u_{1i}^*(t), & u'^{\,*}_{\,1i}(t) \\ u_{2i}^*(t), & u'^{\,*}_{\,2i}(t) \end{pmatrix}, \quad a_{i-1} < t < a_i,$$

$$\sigma_i(t) = \begin{pmatrix} \sigma_{1i}(t), & \sigma'_{1i}(t) \\ \sigma_{2i}(t), & \sigma'_{2i}(t) \end{pmatrix}, \quad t = (a_i + a_{i+1})/2 = a_i^*, \quad i = \overline{1,4}, \tag{2.4}$$

$$\Theta_i^{\pm}(t) = \vartheta_i^{\pm}\, \theta_i^*(t), \quad a_{i-1} < t < a_i,$$

$$\Theta_\infty(t) = \begin{pmatrix} u_{1\infty}(t), & u'_{1\infty}(t) \\ u_{2\infty}(t), & u'_{2\infty}(t) \end{pmatrix},$$

where for $\alpha_{1i} - \alpha_{2i} \neq 0$, $n = 0, 1, 2$, are defined by

$$\vartheta_i^* = \begin{pmatrix} \exp(\pm i\pi\alpha_{1i}), & 0 \\ 0, & \exp(\pm i\pi\alpha_{2i}) \end{pmatrix}, \tag{2.5}$$

and for $\alpha_{1i} - \alpha_{2i} = n$, $n = 0, 1, 2$, by the equations:

$$\vartheta_i^{\pm} = \exp[\pm i\pi\alpha_{2i}] \begin{pmatrix} 1, & 0 \\ \pm i\pi, & 1 \end{pmatrix}, \quad n = 0, 2,$$

$$\vartheta_i^{\pm} = \exp[\pm i\pi\alpha_{2i}] \begin{pmatrix} -1, & 0 \\ \mp\pi i, & 1 \end{pmatrix}, \quad n = 1.$$

It is important to remark [18–21] that the rows $u_{ki}(t)$, $k = 1, 2$; $i = \overline{1,4}$, converge slowly, which makes the calculation process rather difficult.

Let us replace the series $u_{ki}(t)$, $k = 1, 2$, $i = \overline{1,6}$, by the rapidly and uniformly converging functional series [18–21]:

$$u_{ki}(t) = (t - a_i)^{\alpha_{ki}} \widetilde{u}_{ki}(t - a_i),$$

$$\widetilde{u}_{ki}(t - a_i) = 1 + \sum_{n=1}^{\infty} \gamma_{ni}^k (t - a_i), \quad k = 1, 2; \quad i = \overline{1,4}, \tag{2.6}$$

$$u_{n\infty}(t) = t^{-\alpha_{k\infty}} \left(1 + \sum_{n=1}^{\infty} \gamma_{n\infty}^k(t)\right), \tag{2.7}$$

where γ_{ni}^k, $\gamma_{n\infty}^k$ are defined by$f_{ni}(\alpha_i)$ and $f_{k\infty}(\alpha_i)$ as follows:

$$f_{ni}\big[(t-a_i),\beta_k\big] = \alpha_{ki}\, p_{ni}(t-a_i) + q_{ni}(t-a_i), \tag{2.8}$$

$$p_{ni}(t-a_i) = (-1)^{n-1}\sum_{k=1,\,k\neq i}^{5}\beta_i\left(\frac{t-a_i}{a_i-a_k}\right)^n, \quad n=1,2,$$

$$q_{1i}(t-a_i) = c_1(t-a_i),$$

$$q_{ni}(t-a_i) = (-1)^{n-2}\sum_{k=1,k\neq i}^{5}\big[\alpha_{1k}\alpha_{2k}(n-1) + c_k(a_i-a_n)\big]\left(\frac{t-a_i}{a_i-a_k}\right)^n,$$

$$\left|\frac{t-a_i}{a_i-a_k}\right| < 1, \quad k\neq i, \quad \beta_{n\infty}(t) = \sum_{k=1}^{5}\beta_k(a_k/t)^n,$$

$$q_{n\infty} = \sum_{k=1}^{5}\big[\alpha_{1i}\alpha_{2i}(n+1) + c_k a_k\big]\left(\frac{a_k}{t}\right)^n, \quad n=0,1,2,\dots.$$

The local matrix $\Theta_i^-(t)$ is complex-conjugate with respect to the matrix of $\Theta_i^+(t)$. The real matrices $\Theta_i(t)$, $\Theta_i^*(t)$ are the local solutions of a system of equations near the points $t=a_{i-1}$, $t>a_{i-1}$, $t=a_i$, $t<a_i$.

Assume that the elements of these matrices converge to some part of the interval $a_{i-1}<t<a_1$, where the matrices $\Theta_i^*(t)$ and $\Theta_{i-1}(t)$ are related by the following matrix identities [16–21]

$$\Theta_i^*(t) = T_{i-1}\,\Theta_{i-1}(t), \quad i=5,4,3,2, \tag{2.9}$$
$$\Theta_1^*(t) = T_{-\infty}\,\Theta_{-\infty}(t),$$

from which we uniquely define the matrix T_i, $i=\overline{1,5}$. Assume that the convergence domains of the matrices $\Theta_i^*(t)$, $\Theta_{i-1}(t)$ do not intersect. In that case, at the point $t=a_i^*=(a_{i-1}+a_i)/2$ we construct the matrix $\sigma_i(t)$ which converges in the interval $a_{i-1}<t<a_i$. Then it is obvious that we can always pass from the matrix $\Theta_i^*(t)$ to the matrix $\Theta_{i-1}(t)$ in the following sequence:

$$\Theta_i^*(t) = T_{a_i^*}\,\sigma_i(t), \tag{2.10}$$
$$\sigma_i(t) = T_{i-1}^*\,\Theta_{i-1}(t). \tag{2.11}$$

From (2.10) and (2.11) we uniquely define $T_{a_i^*}$ and T_{i-1}^*. Hence it follows that $T\cdot\Theta_5(t)$ can be analytically continued along $-\infty<t<+\infty$.

To define the functions $\omega'(t)$ and $z'(t)$ in the interval $(-\infty,+\infty)$ we consider the matrices [16–21],

$$\chi(t) = T\,\Theta_5(t), \quad \Theta_5(t) = \overline{\Theta_5(t)}, \quad t>a_5,$$

where the matrix T is defined by (2.3). From (2.18) it follows that $T = \overline{T}$ and thus T is a real matrix.

The matrices $\chi^{\pm}(t)$ are solutions of (2.1), where the signs + and − denote respectively the limiting values of the matrix $\chi(t)$ from the upper $(I_m(\zeta) > 0)$ and lower $(I_m(\zeta) < 0)$ half-planes of the plane ζ.

In the sequel, we will define the matrix $\chi^+(t)$, whereas the matrix $\chi^-(t)$ will be defined from the matrix $\chi^+(t)$ by using the formula $\chi^-(t) = \overline{\chi_m^+(t)}$. The following notation will be used:

$$\chi^+(t) = \chi(t), \quad \vartheta_i^+ = \vartheta_i, \quad i = \overline{1,6},$$
$$\chi(t) = T\vartheta_5\Theta_5^*(t),\ a_4 < t < a_5,\ \chi(t) = TT_{+\infty}\Theta_{+\infty}(t),\ a_5 < t < +\infty,$$
$$\chi(t) = T\vartheta_5T_4\Theta_4(t),\ a_4 < t < a_5,\ \chi(t) = T\vartheta_5T_4\vartheta_4\Theta_4^*(t),\ a_3 < t < a_4,$$
$$\chi(t) = T\vartheta_5T_4\vartheta_4T_3\Theta_3(t), \quad a_3 < t < a_4, \tag{2.12}$$
$$\chi(t) = T\vartheta_5T_4\vartheta_4T_3\vartheta_3\Theta_3^*(t), \quad a_2 < t < a_3,$$
$$\chi(t) = T\vartheta_5T_4\vartheta_4T_3\vartheta_3T_2\Theta_2(t), \quad a_2 < t < a_3,$$
$$\chi(t) = T\vartheta_5T_4\vartheta_4T_3\vartheta_3T_2\vartheta_2\Theta_2^*(t),\ a_1 < t < a_2,\ \vartheta_3 = E,\ T_{32} = T_3T_2,$$
$$\chi(t) = T\vartheta_5T_4\vartheta_4T_3\vartheta_3T_2\vartheta_2T_1\Theta_1(t), \quad a_1 < t < a_2,$$
$$\chi(t) = T\vartheta_5T_4\vartheta_4T_3\vartheta_3T_2\vartheta_2T_1\vartheta_1\Theta_1^*(t), \quad -\infty < t < a_1,$$
$$\chi(t) = T\vartheta_5T_4\vartheta_4T_{32}\vartheta_2T_1\vartheta_1T_{-\infty}\Theta_\infty(t),\ T_{32} = T_3T_2,\ -\infty < t < a_1,$$
$$\Theta_5^*(t) = T_4\vartheta_4(t), \quad a_4 < t < a_5, \tag{2.13}$$
$$\Theta_4^*(t) = T_3\vartheta_3(t), \quad a_3 < t < a_4, \tag{2.14}$$
$$\Theta_3^*(t) = T_2\vartheta_2(t), \quad a_2 < t < a_3, \tag{2.15}$$
$$\Theta_2^*(t) = T_1\vartheta_1(t), \quad a_1 < t < a_2, \tag{2.16}$$
$$\Theta_1^*(t) = T_{-\infty}\vartheta_\infty(t), \quad -\infty < t < a_1. \tag{2.17}$$

From system (2.13)–(2.16) we define the matrices T_i, $i = \overline{1,4}$, which depend on the parameters a_i, c_i, $i = \overline{1,5}$.

Substituting successively the matrices $\chi(t)$, $\overline{\chi(t)}$ defined respectively in the intervals (a_{i-1}, a_i), $i = 5,4,3,2,1$, $a_0 = t = -\infty$, and then performing right-multiplication by $[\Theta_i^*(t)]^{-1}$, $i = 5,4,3,2,1$, we obtain the following system of matrix equations [16–21]

$$T\vartheta_5 = g_4T\,\overline{\vartheta_5}, \tag{2.18}$$
$$T\vartheta_5T_4\vartheta_4 = g_3T\,\overline{\vartheta_3}T_4\vartheta_4, \tag{2.19}$$
$$T\vartheta_5T_4\vartheta_4T_{32}\vartheta_2 = g_1T\,\overline{\vartheta_5}T_4\overline{\vartheta_4}T_{32}\overline{\vartheta_2}, \tag{2.20}$$
$$T\vartheta_5T_4\vartheta_4T_{32}\vartheta_2T_1\vartheta_1 = T\,\overline{\vartheta_5}T_4\overline{\vartheta_4}T_{32}\overline{\vartheta_2}T_1\overline{\vartheta_1}. \tag{2.21}$$

The matrices ϑ_i, $i = \overline{1,5}$ are defined as follows:

$$\vartheta_1 = \begin{pmatrix} -i, & 0 \\ 0, & -\exp(i\pi\beta) \end{pmatrix},\ \vartheta_2 = \begin{pmatrix} i\exp(-i\pi\beta) & 0 \\ 0, & 1 \end{pmatrix},\ \vartheta_3 = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

$$\vartheta_4 = \begin{pmatrix} -i, & 0 \\ 0, & \exp(-i\pi\beta) \end{pmatrix}, \ \vartheta_5 = \begin{pmatrix} -i, & 0 \\ 0, & -\exp(i\pi\alpha) \end{pmatrix}. \tag{2.22}$$

From the matrix equations (2.18)–(2.21) we respectively obtain the systems of equations:

$$q = 0, \quad r = 0, \tag{2.23}$$

$$q_4 = 0, \tag{2.24}$$

$$p/s = \big[r_4 \cos(\pi\alpha)\big]/p_4, \tag{2.25}$$

$$r_4 q_{32} \cos\big[\pi(\alpha+\beta)\big] - s_4 s_{32} \sin(\pi\beta) = 0, \tag{2.26}$$

$$r_4 p_{32} \sin(\pi\alpha) + s_4 r_{32} = 0, \tag{2.27}$$

$$p_{32} p_1 \sin(\pi\beta) + q_{32} r_1 = 0, \tag{2.28}$$

$$q_1 p_{32} + s_1 q_{32} \sin(\pi\beta) = 0, \tag{2.29}$$

where

$$T_{32} = T_3 T_2. \tag{2.30}$$

We have obtained two equations for each singular point $t = a_i$, $i = 1, 2, 3, 5,$, and only one equation (1.37) for the point $t = a_3$ – as a result we have a system of nine equations. In addition to these equations, we have obtained a system of three equations $M_k = 0$, $i = 1, 2, 3$.

Systems (2.26), (2.27) and (2.28), (2.29) are homogeneous with respect to (r_4, s_4) and (p_{32}, q_{32}), respectively. For them to be compatible, the following conditions must be fulfilled:

$$p_{32} s_{22}/[r_{32} q_{32}] = -\cos\pi(\alpha+\beta)/[\sin(\pi\alpha)\sin(\pi\beta)], \tag{2.31}$$

$$p_1 s_1/[r_1 q_1] = 1/\sin^2(\pi\beta). \tag{2.32}$$

Equalities (2.31) and (2.32) are related to double or anharmonic relations of four points of one circle.

To determine the parameters a, b, c_k, $k = \overline{1,5}$, we have system (2.24), (2.26), (2.27), (2.28), (2.29), (1.16), (1.47), (1.48), (1.37). As has been shown above, system (1.16), (1.47) and (1.48) is equivalent to the system $M_k = 0$, $k = 1, 2, 3$ [19–21]. From system (1.16), (1.47), (1.48), (1.37) we define the parameters c_1, c_2, c_4, c_5 depending on a, b, c_3, and substitute them into (2.24)–(2.29). Now, a, b, c_3 are defined by system (2.24), (2.26), (2.27), (2.28) and (2,29) [16–21].

From system (2.13)–(2.16) we define the elements of the matrices T_i, $k = \overline{1,4}$, and substitute them into system (2.24)–(2.32).The number of equations is 2 units greater than the number of the unknown parameters. From system (2.24), (2.26)–(2.38) we choose system (2.24), (2.31) and (2.32). If in this system we find the parameters a, b, c_3 and substitute them into (2.25)–(2.29), then from (2.25) we obtain p/s, $ps - rq \neq 0$, and the remaining equations (2.26)–(2.29) must be satisfied identically. It remains to define the parameter s.

At this stage, we must define $\omega'(\zeta)$, $z'(\zeta)$, $w(\zeta)$ and then $\omega(\zeta)$ and $z(\zeta)$. For this, we need to define the linearly independent solutions $u_1(\zeta)$ and $u_2(\zeta)$ of equation (1.13), but to do so we must first define the matrix $x(t)$, we have:

$$
\begin{aligned}
\chi(t) &= \begin{pmatrix} p, & 0 \\ 0, & s \end{pmatrix} \begin{pmatrix} u_{15}(t), & u'_{15}(t) \\ u_{25}(t), & u'_{25}(t) \end{pmatrix}, \quad t > a_5; \\
\chi(t) &= \begin{pmatrix} -ip, & 0 \\ 0, & -se^{i\pi\alpha} \end{pmatrix} \begin{pmatrix} u^*_{15}(t), & u'^*_{15}(t) \\ u^*_{25}(t), & u'^*_{25}(t) \end{pmatrix}, \quad t < a_5; \\
\chi(t) &= \begin{pmatrix} -ipp_4, & 0 \\ -sr_4\exp(i\pi\alpha), & -ss_4\exp(i\pi\alpha) \end{pmatrix} \begin{pmatrix} u_{14}(t), & u'_{14}(t) \\ u_{24}(t), & u'_{24}(t) \end{pmatrix}, \ t > a_4; \\
\chi(t) &= \begin{pmatrix} -pp_4, & 0 \\ isr_4\exp(i\pi\alpha), & -ss_4 \end{pmatrix} \begin{pmatrix} u^*_{14}(t), & u'^*_{14}(t) \\ u^*_{24}(t), & u'^*_{24}(t) \end{pmatrix}, \quad t < a_4; \qquad (2.33) \\
\chi(t) &= \begin{pmatrix} -pp_4p_3, & -pp_4q_3 \\ isr_4p_3\exp(i\pi\alpha) - ss_4r_3, & isr_4q_3\exp(i\pi\alpha) - ss_4s_3 \end{pmatrix} \\
&\quad \times \begin{pmatrix} u_{13}(t), & u'_{13}(t) \\ u_{23}(t), & u'_{23}(t) \end{pmatrix}, \quad t > a_3; \\
\chi(t) &= -pp_4 \begin{pmatrix} p_{32}, & q_{32} \\ -ip_{32}, & \exp(-i\pi\beta)q_{32}/\sin(\pi\beta) \end{pmatrix} \\
&\quad \times \begin{pmatrix} u_{12}(t), & u'_{12}(t) \\ u_{22}(t), & u'_{22}(t) \end{pmatrix}, \quad t > a_2; \\
\chi(t) &= \big(-pp_4\exp(-i\pi\beta)\big) \begin{pmatrix} ip_{32}, & q_{32}\exp(i\pi\beta) \\ p_{32}, & q_{32}/\sin(\pi\beta) \end{pmatrix} \\
&\quad \times \begin{pmatrix} u^*_{12}(t), & u'^*_{12}(t) \\ u^*_{22}(t), & u'^*_{22}(t) \end{pmatrix}, \quad t < a_2; \\
\chi(t) &= \big(-pp_4\exp(-i\pi\beta)\big) \\
&\quad \times \begin{pmatrix} ip_{32}p_1\cos(\pi\beta)\exp(i\pi\beta), & q_{32}s_1\cos(\pi\beta) \\ 0, & q_{32}s_1\cos^2(\pi\beta)/\sin(\pi\beta) \end{pmatrix} \\
&\quad \times \begin{pmatrix} u_{11}(t), & u'_{11}(t) \\ u_{21}(t), & u'_{21}(t) \end{pmatrix}, \quad t > a_1; \\
\chi(t) &= \big(-pp_4\cos(\pi\beta)\exp(-i\pi\beta)\big) \begin{pmatrix} p_{32}p_1, & -q_{32}s_1 \\ 0, & -q_{32}s_1\cos(\pi\beta)/\sin(\pi\beta) \end{pmatrix} \\
&\quad \times \begin{pmatrix} u^*_{11}(t), & u'^*_{11}(t) \\ u^*_{21}(t), & u'^*_{21}(t) \end{pmatrix}, \quad t < a_1.
\end{aligned}
$$

From (2.33) we can define $u_1(t)$ and $u_2(t)$ in the interval $(-\infty < t < +\infty)$, we have:

$$
\begin{aligned}
&u_1(t) = pu_{15}(t), \quad u_2(t) = su_{25}(t), \quad t > a_5;\\
&u_1(t) = -ipu^*_{15}(t), \quad u_2(t) = -s\exp(i\pi\alpha)u^*_{25}(t), \quad t < a_5;\\
&u_1(t) = -ipp_4u_{14}(t),\\
&u_2(t) = -sr_4\exp(i\pi\alpha)u_{14}(t) - ss_4\exp(i\pi\alpha)u_{24}(t), \quad t > a_4;\\
&u_1(t) = -ipp_4u^*_{14}(t), \;\; u_2(t) = isr_4\exp(i\pi\alpha)u^*_{14}(t) - ss_4u^*_{24}, \;\; t < a_4;\\
&u_1(t) = -pp_4\big(p_3u_{13}(t) + q_3u_{23}(t)\big), \quad t > a_3;\\
&u_2(t) = s\big[ir_4p_3\exp(i\pi\alpha) - s_4r_3\big]u_{13}(t)+\\
&\qquad\quad + s\big[ir_4q_3\exp(i\pi\alpha) - s_4s_3\big]u_{23}(t), \quad t > a_3;\\
&u_1(t) = -pp_4\big[p_{32}u_{12}(t) + q_{32}u_{22}(t)\big], \quad t > a_2; \qquad\qquad (2.34)\\
&u_2(t) = pp_4\big[ip_{32}u_{12}(t) - q_{32}\exp(-i\pi\beta)/[\sin(\pi\beta)]u_{22}(t)\big], \quad t > a_2;\\
&u_1(t) = -pp\big[ip_{32}\exp(-i\pi\beta)u^*_{12}(t) + q_{32}u^*_{22}(t)\big], \quad t < a_2;\\
&u_2(t) = -pp_4\big[p_{32}\exp(-i\pi\beta)u^*_{12}(t)\\
&\qquad\quad + q_{32}\exp(-i\pi\beta)/\sin(\pi\beta)u^*_{22}(t)\big], \; t < a_2;\\
&u_1(t) = -pp_4\cos(\pi\beta)\big[ip_{32}p_1u_{11}(t) + q_{32}s_1\exp(-i\pi\beta)u_{21}(t)\big], \; t > a_1;\\
&u_2(t) = -pp_4q_{32}p_1\exp(-i\pi\beta)\cos^2(\pi\beta)/\sin(\pi\beta)u_{21}(t), \quad t > a_1;\\
&u_1(t) = -pp_4\cos(\pi\beta)\exp(-i\pi\beta)\big[p_{32}p_2u^*_{11}(t) - q_{32}s_1u^*_{21}(t)\big], \; t < a_1;\\
&u_2(t) = pp_4q_{32}s_1\cos^2(\pi\beta)\exp(-i\pi\beta)/\sin(\pi\beta)u^*_{21}(t), \quad t < a_1.
\end{aligned}
$$

After defining $u_1(t)$ and $u_2(t)$, we define $\omega(\zeta)$ and $z(\zeta)$. The components of the vector $\Phi'(t)$ along the t-axis are defined to within the sign as follows:

$$\omega'(t) = \chi_1(t)\,u_1(t), \;\; z'(t) = \chi_1(t)\,u_2(t), \;\; -\infty < t < +\infty. \qquad (2.35)$$

Equalities (2.35) can be rewritten as

$$d\omega(t) = \chi_1(t)\,u_1(t)dt, \; dz(t) = \chi_1(t)\,u_2(t)dt, \; -\infty < t < +\infty. \qquad (2.36)$$

The integration of (2.36) gives

$$\omega(t) = \int_{-\infty}^{t} \chi_1(t)\,u_1(t)dt + \omega(-\infty); \qquad (2.37)$$

$$z(t) = \int_{-\infty}^{t} \chi_1(t)\,u_2(t)dt + z(-\infty); \qquad (2.38)$$

$$\omega(t) = \int_{a_i}^{t} \chi_1(t)\,u_1(t)dt + \omega(a_i + 0); \qquad (2.39)$$

$$z(t) = \int_{a_i}^{t} \chi_1(t)\,u_1(t)dt + z(a_i + 0), \qquad (2.40)$$

where $\omega(a_i+0)$, $z(a_i+0)$ denote the limiting values of these functions at the points $t=a_i$, $i=\overline{1,5}$ on the right.

It is obvious that the functions $\omega(t)$, $z(t)$ defined by (2.37)–(2.40) will satisfy the boundary conditions (1.2) because the vector $f(t)$ is piecewise-constant [18–21].

Separating the real parts from the imaginary ones in (2.37)–(2.40), we obtain the expressions for the functions $\varphi(t)$, $\psi(t)$, $x(t)$ and $y(t)$. Moreover, if we take $t=a_1$ in (2.37) and (2.38) and $t=a_{i+1}$ in (2.39) and (2.40), then we obtain

$$\omega(a_1-0)=\int_{-\infty}^{a_1}\chi_1(t)\,u_1(t)dt+\omega(-\infty), \tag{2.41}$$

$$z(a_1-0)=\int_{-\infty}^{a_1}\chi_1(t)\,u_1(t)dt+z(-\infty), \tag{2.42}$$

$$\omega(a_{i+1}-0)=\int_{a_i}^{a_{i+1}}\chi_1(t)\,u_1(t)dt+\omega(a_i+0),\quad i=\overline{1,5}, \tag{2.43}$$

$$z(a_{i+1}-0)=\int_{a_i}^{a_{i+1}}\chi_1(t)\,u_2(t)dt+z(a_i+0),\quad i=\overline{1,5}. \tag{2.44}$$

For the interval (a_5,e_6) we must take into account that $\chi_1(t)=i\chi_1^*(t)$.

Upon the integration of (2.34), for the intervals (a_i,a_{i+1}), $i=1,2,3,4$, we respectively have:

$$\begin{aligned}\omega(a_1^*-0)-\omega(a_1+0)&=-pp_4\cos(\pi\beta)\int_{a_1}^{a_1^*}\chi_1(t)\big[ip_{32}p_1u_{11}(t)\\&\quad+q_{32}s_1\exp(-i\pi\beta)u_{21}(t)\big]dt, &(2.45)\\ z(a_1^*-0)-z(a_1+0)&=-pp_4q_{32}p_1\cos^2(\pi\beta)\\&\quad\times\exp(-i\pi\beta)/\sin(\pi\beta)\int_{a_1}^{a_1^*}\chi_1(t)u_{21}(t)dt, &(2.46)\\ \omega(a_2-0)-\omega(a_1^*+0)&=\\&\quad-pp_4\int_{a_1^*}^{a_2}\chi_1(t)\big[ip_{32}\exp(-i\pi\beta)u_{12}^*(t)+q_{32}u_{22}^*(t)\big]dt, &(2.47)\\ z(a_2-0)-z(a_1^*+0)&=-pp_4\exp(-i\pi\beta)\int_{a_1^*}^{a_2}\chi_1(t)\big[p_{32}u_{12}^*(t)\\&\quad+(q_{32}/\sin(\pi\beta))u_{22}^*(t)\big]dt, &(2.48)\\ \omega(a_2^*-0)-\omega(a_2+0)&=-pp_4\int_{a_2}^{a_2^*}\big[p_{32}u_{12}(t)+q_{32}u_{22}(t)\big]\chi_1(t)dt, &(2.48_1)\\ z(a_2^*-0)-z(a_2+0)&=pp_4\int_{a_2}^{a_2^*}\chi_1(t)\big[ip_{32}u_{12}(t)\end{aligned}$$

$$- q_{32} \exp(-i\pi\beta)/\sin(\pi\beta) u_{22}(t)\big] dt, \tag{2.49}$$

$$\omega(a_3 - 0) - \omega(a_2^* + 0) = -pp_4 \int_{a_2^*}^{a_3} \chi_1(t)\big[p_3 u_{13}(t) + q_3 u_{23}(t)\big] dt, \tag{2.50}$$

$$z(a_3 - 0) - z(a_2^* + 0) = s\big[ir_4 p_3 \exp(i\pi\alpha) - s_4 r_3\big] \int_{a_2^*}^{a3} \chi_1(t) u_{13}(t) dt$$

$$+ s\big[ir_4 q_3 \exp(i\pi\alpha) - s_4 s_3\big] \int_{a_2^*}^{a3} \chi_1(t) u_{23}(t) dt, \tag{2.51}$$

$$\omega(a_4 - 0) - \omega(a_3 + 0) = pp_4 \int_{a_3}^{a_4} \chi_1(t) u_{14}^*(t) dt, \tag{2.52}$$

$$z(a_4 - 0) - z(a_3 + 0) = s \int_{a_3}^{a_4} \chi_1(t)$$

$$\times \big[ir_4 \exp(i\pi\alpha) u_{14}^*(t) - s_4 u_{24}^*(t)\big] dt, \tag{2.53}$$

$$\omega(a_4 - 0) - \omega(a_4 + 0) = -pp_4 \int_{a_3}^{a_4^*} \chi_1(t) u_{14}(t) dt, \tag{2.54}$$

$$z(a_4^* - 0) - z(a_4 + 0) = (-1) s \exp(i\pi\alpha)$$

$$\times \int_{a_4}^{a_4^*} \chi_1(t)\big[r_4 u_{14}(t) + s_4 u_{24}(t)\big] dt, \tag{2.55}$$

$$\omega(a_5 - 0) - \omega(a_4^* + 0) = -ip \int_{a_4^*}^{a_5} \chi_1(t) u_{15}^*(t) dt, \tag{2.56}$$

$$z(a_5 - 0) - z(a_4^* + 0) = -s \exp(i\pi\alpha) \int_{a_4^*}^{a_5} \chi_1(t) u_{25}^*(t) dt, \tag{2.57}$$

$$\omega(e_6 - 0) - \omega(a_5 - 0) = pi \int_{a_5}^{e_6} \chi_1^*(t) u_{15}(t) dt, \tag{2.58}$$

$$z(e_6 - 0) - z(a_5 + 0) = si \int_{a_5}^{e_6} \chi_1^*(t) u_{25}(t) dt. \tag{2.59}$$

Based on formulas (2.45)–(2.59), we consider the sums

$$\begin{aligned} \omega(a_i^* - 0) - &\omega(a_i + 0) + \omega(a_{i+1} - 0) - \omega(a_i^* + 0) \\ &= \omega(a_{i+1} - 0) - \omega(a_i + 0), \end{aligned} \tag{2.60}$$

$$\begin{aligned} z(a_i^* - 0) - &z(a_i + 0) + z(a_{i+1} - 0) - z(a_i^* + 0) \\ &= z(a_{i+1} - 0) - z(a_i + 0), \quad i = \overline{1,5}. \end{aligned} \tag{2.61}$$

Thus we obtain

$$\omega(a_2 - 0) - \omega(a_1 + 0) = -pp_4 \cos(\pi\beta) \int_{a_1}^{a_1^*} \chi_1(t)\big[ip_{32} p_1 u_{11}(t)$$

$$+ q_{32} s_1 \exp(-i\pi\beta) u_{21}(t)\big] dt -$$

$$- pp_4 \int_{a_1}^{a_2} \chi_1(t)\big[ip_{32}\exp(-i\pi\beta)u_{12}^*(t) + q_{32}u_{22}^*(t)\big]dt, \tag{2.62}$$

$$z(a_2-0) - z(a_1+0) = -pp_4 q_{32} p_1 \cos^2(\pi\beta)\exp(-i\pi\beta)/\sin(\pi\beta)$$
$$\times \int_{a_1}^{a_1^*} \chi_1(t)u_{21}(t)dt + (-1)pp_4\exp(-\pi\beta)$$
$$\times \int_{a_1^*}^{a_2} \chi_1(t)\big[p_{32}u_{12}^*(t) + (q_{32}/\sin(\pi\beta))u_{22}^*(t)\big]dt, \tag{2.63}$$

$$\omega(a_3-0) - \omega(a_2+0) = -pp \int_{a_2}^{a_2^*} \chi_1(t)\big[p_{32}u_{12}(t) - q_{32}u_{22}(t)\big]dt$$
$$- pp_4 \int_{a_2^*}^{a_3} \chi_1(t)\big[p_3u_{13}(t) + q_3u_{23}(t)\big]dt, \tag{2.64}$$

$$z(a_3-0) - z(a_2+0) = pp_4 \int_{a_2}^{a_2^*} \chi_1(t)\big[ip_{32}u_{12}(t)$$
$$+ q_{32}\exp(-i\pi\beta)/\sin(\pi\beta)u_{22}(t)\big]dt$$
$$+ s\big[ir_4p_3\exp(i\pi\alpha) - s_4r_3\big] \int_{a_2^*}^{a_3} \chi_1(t)u_{13}dt$$
$$+ s\big[ir_4q_3\exp(i\pi\alpha) - s_4s_3\big] \int_{a_2^*}^{a_3} \chi_1(t)u_{23}dt, \tag{2.65}$$

$$\omega(a_5-0) - \omega(a_4+0) = -ipp_4 \int_{a_4}^{a_4^*} \chi_1(t)u_{14}(t)dt$$
$$- ip \int_{a_4^*}^{a_5} \chi_1(t)u_{15}^*(t)dt, \tag{2.66}$$

$$z(a_5-0) - z(a_4+0) = (-1)s\exp(i\pi\alpha) \int_{a_4}^{a_4^*} \chi_1(t)\big[r_4u_{14}(t) + s_4u_{24}(t)\big]dt$$
$$- s\exp(i\pi\alpha) \int_{a_4^*}^{a_5} \chi_1(t)u_{25}^*(t)dt. \tag{2.67}$$

Separating the real parts from the imaginary ones in (2.52), (2.53), (2.58), (2.62)–(2.67), we get:

$$y(a_2) = pp_4\Big\{q_{32}s_1\cos^2(\pi\beta) \int_{a_1}^{a_1^*} \chi_1(t)u_{21}(t)dt$$
$$+ \int_{a_1^*}^{a_2} \chi_1(t)\big[p_{32}\sin(\pi\beta)u_{12}^*(t) + q_{32}u_{22}^*(t)\big]dt\Big\}, \tag{2.68}$$

$$Q = pp_4\cos(\pi\beta)\Big\{ \int_{a_1}^{a_1^*} \chi_1(t)\big[-p_{32}p_1u_{11}(t) + q_{32}s_1\sin(\pi\beta)u_{21}(t)\big]dt$$

$$-p_{32}\int_{a_1^*}^{a_2}\chi_1(t)u_{12}^*(t)dt\Big\}, \tag{2.69}$$

$$x(a_2)=L-pp_4\cos(\pi\beta)\Big\{q_{32}p_1\cos^2(\pi\beta)/\sin(\pi\beta)\int_{a_1}^{a_1^*}\chi_1(t)u_{21}(t)dt$$

$$+\int_{a_1^*}^{a_2}\chi_1(t)\big[p_{32}u_{12}^*(t)+(q_{32}/\sin(\pi\beta))u_{22}^*(t)\big]dt\Big\}, \tag{2.70}$$

$$y(a_2)=pp_4\Big\{q_{32}p_1\cos^2(\pi\beta)\int_{a_1}^{a_1^*}\chi_1(t)u_{21}(t)dt$$

$$+\int_{a_1^*}^{a_2}\chi_1(t)\big[\sin(\pi\beta)p_{32}u_{12}^*(t)+q_{32}u_{22}^*(t)\big]dt\Big\}, \tag{2.71}$$

$$y(a_3)-y(a_2)=pp_4\Big\{\int_{a_2}^{a_2^*}\chi_1(t)\big[p_{32}u_{12}(t)+q_{32}u_{22}(t)\big]dt$$

$$+\int_{a_2^*}^{a_2}\chi_1(t)\big[p_3u_{13}(t)+q_3u_{23}(t)\big]dt\Big\}, \tag{2.72}$$

$$x(a_3)-x(a_2)=-pp_4q_{32}\cot(\pi\beta)\int_{a_2}^{a_2^*}\chi_1(t)u_{22}(t)dt$$

$$-s\big[r_4p_3\sin(\pi\alpha)+s_4r_3\big]\int_{a_2^*}^{a_2}\chi_1(t)u_{13}(t)dt$$

$$-s\big[r_4q_3\sin(\pi\alpha)+s_4s_3\big]\int_{a_2^*}^{a_3}\chi_1(t)u_{23}(t)dt, \tag{2.73}$$

$$y(a_3)-y(a_2)=pp_4\Big\{\int_{a_2}^{a_2^*}\chi_1(t)\big[p_{32}u_{12}(t)+q_{32}u_{22}(t)\big]dt$$

$$+sr_4\cos(\pi\alpha)\int_{a_2^*}^{a_3}\chi_1(t)\big[p_3u_{13}(t)+q_3u_{23}(t)\big]dt, \tag{2.74}$$

$$y(a_4)-y(a_3)=pp_4\int_{a_3}^{a_4}\chi_1(t)u_{14}^*(t)dt, \tag{2.75}$$

$$x(a_4)-x(a_3)=-s\int_{a_3}^{a_4}\chi_1(t)\big[r_4\sin(\pi\alpha)u_{14}^*(t)+su_{24}^*(t)\big]dt, \tag{2.76}$$

$$y(a_4)-y(a_3)a_3=sr_4\cos(\pi\alpha)\int_{a_3}^{a_4}\chi_1(t)u_{14}^*(t)dt, \tag{2.77}$$

$$Q-Q^1=pp_4\int_{a_4}^{a_4^*}\chi_1(t)u_{14}(t)dt+\int_{a_4^*}^{a_5}\chi_1(t)u_{15}^*(t)dt, \tag{2.78}$$

$$x(a_4)=s\cos(\pi\alpha)\int_{a_4}^{a_4^*}\chi_1(t)\big[r_4u_{14}(t)+s_4u_{24}(t)\big]dt$$

$$+s\cos(\pi\alpha)\int_{a_4^*}^{a_5}\chi_1(t)u_{25}^*(t)dt, \tag{2.79}$$

$$H - H_1 = s\sin(\pi\alpha)\int_{a_4}^{a_4^*}\chi_1(t)\big[r_4u_{14}(t) + s_4u_{24}(t)\big]dt$$
$$+s\sin(\pi\alpha)\int_{a_4^*}^{a_5}\chi_1(t)u_{25}^*(t)dt, \tag{2.80}$$
$$Q^1 = -p\int_{a_5}^{e_6}\chi_1^*(t)u_{15}(t)dt, \tag{2.81}$$
$$H_1 = -s\int_{a_5}^{e_6}\chi_1^*(t)u_{25}(t)dt. \tag{2.82}$$

It can be immediately verified that the following equalities are fulfilled:

$$y(a_2) = \big[x(a_1) - x(a_2)\big]\tan(\pi\alpha), \tag{2.83}$$
$$H - H_1 = x(a_4)\tan(\pi\alpha). \tag{2.84}$$

From (2.80) we can define the parameter s and substitute it into (2.82), we obtain an equation with respect to e_6. From this equation we can define the parameter e_6. Then we can solve equation (2.80) with respect to the parameter s. Now we can define all the parameters, for example, $y(a_2)$, Q, Q^1 and so on. Thus, using formula (2.40) we can define the parametric equation $x(t)$ and $y(t)$.

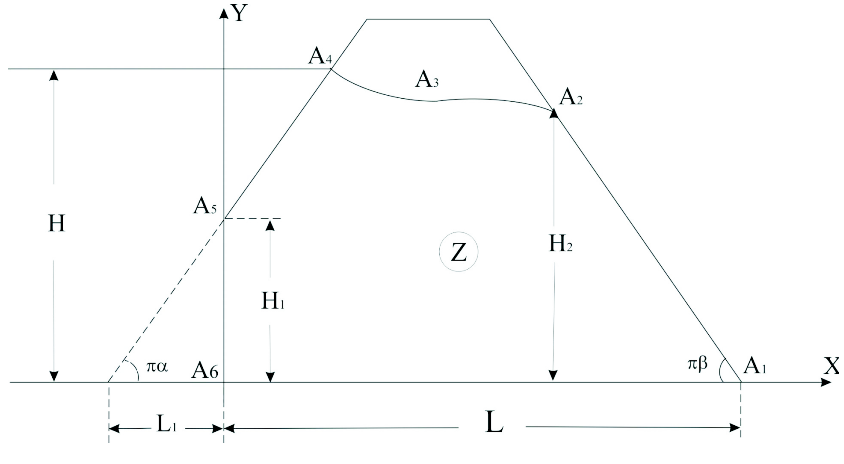

Fig.1

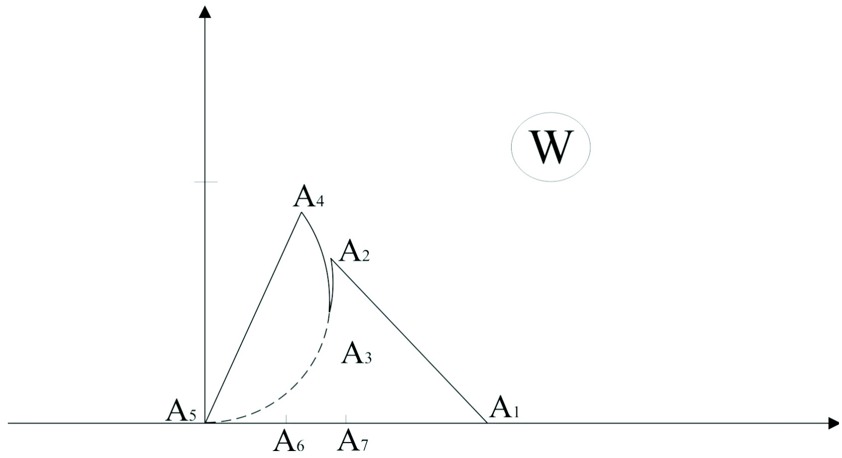

Fig.2

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