NUMERICAL SOLUTION OF SOME PLANE BOUNDARY VALUE PROBLEMS OF THE THEORY OF BINARY MIXTURES BY THE BOUNDARY ELEMENT METHOD

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Abstract

The paper deals with the application of the method of boundary elements to the numerical solution of plane boundary problems in the case of the linear theory of elastic mixtures. First the Kelvin problem is solved analytically when concentrated force is applied to a point in an infinite domain filled with a binary mixture of two isotropic elastic materials. By integrating the solution of this problem we obtain a solution of the problem when constant forces are distributed over an interval segment. The obtained singular solutions are used for applying one of the boundary element methods called the fictitious load method to the solution of various boundary value problems for both finite and infinite domains.

Key words and phrases: Binary mixture, Kelvin problem, boundary element method, fictitious load.

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introduction

In the present paper we consider the Green-Naghdi-Steel model of the linear theory of a mixture of two isotropic elastic materials [1], [2]. The Kelvin problem [3] is solved in the case of plane deformation when concentrated force is applied to a point in an infinite domain filled with a binary mixture. By integrating the solution of this problem we obtain a solution of the problem for the infinite domain when constant forces are distributed over an interval segment. The obtained singular solutions are used for the application of the boundary element method to the numerical solution of various boundary value problems for both finite and infinite domains.

1 Basic relations of the linear theory of elastic mixtures

In the Cartesian system $Ox_1x_2x_3$, the three-dimensional equations of static equilibrium of a body consisting of a mixture of two isotropic elastic materials have the form [4]

$$\begin{cases} \partial_i \sigma'_{ij} - \pi_j + \rho_1 f'_j = 0, \\ \partial_i \sigma''_{ij} + \pi_j + \rho_2 f''_j = 0, \quad i,j = 1,2,3, \end{cases} \tag{1}$$

while the relations of elasticity are written as follows

$$\begin{aligned} \sigma'_{ij} &= \left(-\alpha_2 + \lambda_1 \varepsilon'_{kk} + \lambda_3 \varepsilon''_{kk}\right) \delta_{ij} + 2\mu_1 \varepsilon'_{ij} + 2\mu_3 \varepsilon''_{ij} - 2\lambda_5 h_{ij}, \\ \sigma''_{ij} &= \left(\alpha_2 + \lambda_4 \varepsilon'_{kk} + \lambda_2 \varepsilon''_{kk}\right) \delta_{ij} + 2\mu_3 \varepsilon'_{ij} + 2\mu_2 \varepsilon''_{ij} + 2\lambda_5 h_{ij}, \end{aligned} \tag{2}$$

where σ'_{ij}, σ''_{ij} are the partial stress tensor components, $\pi_j \equiv \partial_j \pi$ and

$$\pi = \frac{\alpha_2 \rho_2}{\rho} \varepsilon'_{kk} + \frac{\alpha_2 \rho_1}{\rho} \varepsilon''_{kk}, \quad \rho = \rho_1 + \rho_2,$$

$\rho_1 > 0$, $\rho_2 > 0$ are the densities of the mixture components; f'_j, f''_j are the components of the mass force vectors; α_2, λ_1, λ_2, λ_3, λ_4, λ_5, μ_1, μ_2, μ_3 are elasticity constants; $\alpha_2 = \lambda_3 - \lambda_4$; $\epsilon'_{ij} = \epsilon'_{ji}$, $\epsilon''_{ij} = \epsilon''_{ji}$ are the partial deformation tensor components

$$\varepsilon'_{ij} = \frac{1}{2}\left(\partial_i u'_j + \partial_j u'_i\right), \quad \varepsilon''_{ij} = \frac{1}{2}\left(\partial_i u''_j + \partial_j u''_i\right), \tag{3}$$

$h_{ij} = -h_{ji}$ are the rotation tensor components

$$h_{ij} = \frac{1}{2}\left(\partial_i u'_j - \partial_j u'_i + \partial_j u''_i - \partial_i u''_j\right), \tag{4}$$

$u' = (u'_1, u'_2, u'_3), u'' = (u''_1, u''_2, u''_3)$ are the partial displacement vectors; $\partial_j = \dfrac{\partial}{\partial x_j}$.

The Latin indexes take values 1, 2, 3. Summation is assumed to be performed over the repeated indexes.

For the sake of simplicity we introduce the following notation (column-matrices)

$$P'_{ij} := \sigma'_{ij} - \delta_{ij}\left(\pi - \alpha_2\right), \quad P''_{ij} := \sigma''_{ij} + \delta_{ij}\left(\pi - \alpha_2\right), \tag{5}$$

$$\sigma_{ij} := \left(P'_{ij}, P''_{ij}\right)^T, \quad u_j := \left(u'_j, u''_j\right)^T, \quad \mathrm{e}_{ij} := \left(\varepsilon'_{ij}, \varepsilon''_{ij}\right)^T,$$
$$\hbar_{ij} := \left(h_{ij}, h_{ji}\right)^T. \tag{6}$$

Using notation (5), (6), we rewrite relations (1) − (4) as follows

$$\partial_i \sigma_{ij} + \varphi_j = 0, \tag{7}$$

$$\sigma_{ij} = \Lambda \mathrm{e}_{kk} \delta_{ij} + 2M \mathrm{e}_{ij} - 2\lambda_5 \hbar_{ij}, \tag{8}$$

where $\varphi_j = (\rho_1 f'_j, \rho_2 f''^{j})^T$,

$$\Lambda := \begin{pmatrix} \lambda_1 - \dfrac{\alpha_2 \rho_2}{\rho} & \lambda_3 - \dfrac{\alpha_2 \rho_1}{\rho} \\ \lambda_4 + \dfrac{\alpha_2 \rho_2}{\rho} & \lambda_2 + \dfrac{\alpha_2 \rho_1}{\rho} \end{pmatrix}, \qquad M := \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix};$$

$$\Theta_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right), \quad \hbar_{ij} = \frac{1}{2} S \left(\partial_i u_j - \partial_j u_i\right), \quad S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{9}$$

With (9) taken into account, relation (8) can also be written as follows

$$\sigma_{ij} = \Lambda \partial_k u_k \delta_{ij} + \left(M - \lambda_5 S\right) \partial_i u_j + \left(M + \lambda_5 S\right) \partial_j u_i. \tag{10}$$

By substituting (10) into (7) we obtain equations in terms of displacement vector components

$$\mathrm{A}\Delta \mathrm{u_j} + \mathrm{B}\partial_\mathrm{j}\left(\partial_\mathrm{k} \mathrm{u_k}\right) + \varphi_\mathrm{j} = 0, \tag{11}$$

where

$$\mathrm{A} := M - \lambda_5 S, \quad B := M + \lambda_5 S + \Lambda, \quad \Delta := \partial_k \partial_{k.}$$

Let us consider the case of plane deformation for a cylindrical body when $u_x \equiv u_1$, $u_y \equiv u_2$ and $\varphi_x \equiv \varphi_1$, $\varphi_y \equiv \varphi_2$ do not depend on x_3, and $u_3 = 0$. $\varphi_3 = 0$. Then $\sigma_{xx} \equiv \sigma_{11}$, $\sigma_{yy} \equiv \sigma_{22}$, $\sigma_{xy} \equiv \sigma_{12}$, $\sigma_{yx} \equiv \sigma_{21}$ do not depend on x_3 and, moreover, $\sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = 0$. We have

$$\begin{cases} \sigma_{xx,x} + \sigma_{yx,y} + \varphi_x = 0, \\ \sigma_{xy,x} + \sigma_{yy,y} + \varphi_y = 0, \end{cases}$$

$$\begin{cases} \sigma_{xx} = \Lambda\theta + 2M u_{x,x}, \quad \sigma_{yy} = \Lambda\theta + 2M u_{y,y}, \\ \sigma_{xy} = A u_{y,x} + (B - \Lambda) u_{x,y}, \quad \sigma_{yx} = A u_{x,y} + (B - \Lambda) u_{y,x}, \\ \theta = u_{x,x} + u_{y,y}, \quad (\cdot)_{,x} := \dfrac{\partial(\cdot)}{\partial x}, \quad (\cdot)_{,y} := \dfrac{\partial(\cdot)}{\partial y}, \\ x_1 = x, \quad x_2 = y. \end{cases} \tag{12}$$

Relation (11) can now be written in the form

$$\begin{cases} A\Delta_2 u_x + B\theta_{,x} + \varphi_x = 0, \\ \\ A\Delta_2 u_y + B\theta_{,y} + \varphi_y = 0, \end{cases} \tag{13}$$

where $\Delta_2(\cdot) = (\cdot)_{,xx} + (\cdot)_{,yy}$.

On the Oxy-plane we introduce the complex variable $z = x + iy$. Then system (13) can be written in the complex form

$$4A\left(u_x + iu_y\right)_{,z\bar{z}} + 2B\theta_{,\bar{z}} + \varphi_+ = 0, \tag{14}$$

where $\bar{z} = x - iy$, $(\cdot)_{,z} = \dfrac{1}{2}\left[(\cdot)_{,x} - i(\cdot)_{,y}\right]$, $(\cdot)_{,\bar{z}} = \dfrac{1}{2}\left[(\cdot)_{,x} + i(\cdot)_{,y}\right]$, $\varphi_+ = \varphi_x + i\varphi_y$.

A general solution of system (14) is represented by the following analog of the Kolosov-Muskhelishvili formula [5]:

$$2(u_x + iu_y) = A^*\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{15}$$

where $\varphi(z) = (\varphi_1(z), \varphi_2(z))^T$, $\psi(z) = (\psi_1(z), \psi_2(z))^T$ are column-matrices consisting of arbitrary analytic functions of the complex variable in the considered domain, $A^* = I + 2B^{-1}A$, where I is the 2×2 unit matrix.

For complex combinations of the stress tensor components, by formulas (12) and (15) we obtain the expressions

$$\begin{aligned} &\sigma_{yy} - \sigma_{xx} + i\left(\sigma_{xy} + \sigma_{yx}\right) = 2M\left[\bar{z}\Phi'(z) + \Psi(z)\right], \\ &\sigma_{xx} + \sigma_{yy} + i\left(\sigma_{xy} - \sigma_{yx}\right) = 2\left[\left(A - \lambda_5 SA^*\right)\Phi(z) + M\overline{\Phi(z)}\right], \end{aligned} \tag{16}$$

where

$$\Phi(z) = \left(\varphi_1'(z),\ \varphi_2'(z)\right)^T, \quad \Psi(z) = \left(\psi_1'(z),\ \psi_2'(z)\right)^T.$$

2 The Kelvin problem for a binary mixture in the case of plane deformation

It is assumed that we have an infinite domain with a circular hole of radius R and center at the origin. We consider the case of plane deformation for a binary mixture. Let stresses damp at infinity and stresses of constant value and direction ($z = re^{i\theta}$) [6] be applied to the circular contour:

$$\sigma_{rx} = \frac{1}{2\pi R}F_x, \quad \sigma_{ry} = \frac{1}{2\pi R}F_y, \quad F_x = (F_x', F_x'')^T, \quad F_y = (F_y', F_y'')^T.$$

where F'_x, F'_y, F''_x, F''_y are constant values. On the circular hole contour ($r = R$) the formula

$$\sigma_{rr} - i\sigma_{r\vartheta} = -\frac{1}{2\pi R}\left(F_x - iF_y\right)e^{i\vartheta}$$

is valid for the polar components of the stress tensor. Using the introduced analytic functions the latter boundary condition can be rewritten as follows:

$$\begin{aligned} &M\Phi\left(z\right) + \left(A - \lambda_5 SA^*\right)\overline{\Phi\left(z\right)} - M\left[\overline{z}\Phi'\left(z\right) + \Psi\left(z\right)\right]e^{2i\vartheta} = \\ &= -\tfrac{1}{2\pi R}\left(F_x - iF_y\right)e^{i\vartheta} \quad \text{on} \quad r = R. \end{aligned} \tag{17}$$

Since there are no stresses at infinity, the expansions of the functions $\Phi(z)$ and $\Psi(z)$ into power series do not contain free terms

$$\Phi\left(z\right) = \sum_{n=1}^{\infty}\frac{a_n}{r^n}e^{-in\vartheta}, \quad \Psi\left(z\right) = \sum_{n=1}^{\infty}\frac{b_n}{r^n}e^{-in\vartheta}, \tag{18}$$

where $a_n = (a'_n, a''_n)$, $b_n = (b'_n, b''_n)^T$ are the values we want to define.

Substituting expansions (18) into the boundary conditions (17) and taking into account the condition of displacement uniqueness

$$A^* a_1 + \overline{b_1} = 0,$$

we obtain the coefficient values:

$$a_1 = -\frac{1}{2\pi}\left(I + A^*\right)^{-1}A^{-1}\left(F_x + iF_y\right), \quad b_1 = -A^*\overline{a}_1, \quad b_3 = 2R^2 a_1.$$

All other coefficients are equal to zero. Therefore

$$\Phi\left(z\right) = \frac{a_1}{z}, \qquad \Psi\left(z\right) = \frac{b_1}{z} + \frac{b_3}{z^3}.$$

Substituting the obtained values of $\Phi(z)$ and $\Psi(z)$ into formulas (15) and (16), we obtain the values for the displacement vector and stress tensor components. Let now $R \to 0$, and σ_{rx}, σ_{xy} increase infinitely, but the principal vector remain invariable. Then

$$\Phi\left(z\right) = \frac{a_1}{z}, \qquad \Psi\left(z\right) = \frac{b_1}{z} = -\frac{A^*}{z}\overline{a}_1.$$

In that case, by virtue of formula (15) we obtain

$$\begin{aligned} u_x + iu_y = &-\frac{1}{4\pi}A^*\left(I + A^*\right)^{-1}A^{-1}\left(F_x + iF_y\right)\ln z\overline{z} + \\ &+\frac{1}{4\pi}\left(I + A^*\right)A^{-1}\left(F_x - iF_y\right)\frac{z}{\overline{z}}. \end{aligned} \tag{19}$$

Separating the real and the imaginary parts of formula (19), we have

$$\begin{aligned} u_x &= \left(A^* G - x G_{,x}\right) F_x + \left(-y G_{,x}\right) F_y - \frac{1}{4\pi}\left(I + A^*\right)^{-1} A^{-1} F_x, \\ u_y &= \left(-x G_{,y}\right) F_x + \left(A^* G - y G_{,y}\right) F_y - \frac{1}{4\pi}\left(I + A^*\right)^{-1} A^{-1} F_y, \end{aligned} \tag{20}$$

where the following notation has been introduced:

$$G\left(x, y\right) = -\frac{1}{2\pi}\left(I + A^*\right)^{-1} A^{-1} \ln\left(x^2 + y^2\right)^{1/2}. \tag{21}$$

Discarding in formula (20) the constant values corresponding to a rigid displacement, we obtain

$$\begin{aligned} u_x &= \left(A^* G - x G_{,x}\right) F_x + \left(-y G_{,x}\right) F_y, \\ u_y &= \left(-x G_{,y}\right) F_x + \left(A^* G - y G_{,y}\right) F_y. \end{aligned} \tag{22}$$

Substitution of formulas (22) into formulas (12) gives the expressions for stresses

$$\begin{aligned} \sigma_{xx} &= \left[\left(\Lambda + 2M\right)\left(A^* - I\right) G_{,x} - 2MxG_{,xx}\right] F_x + \\ &\quad + \left[\Lambda\left(A^* - I\right) G_{,y} - 2MyG_{,xx}\right] F_y; \\ \sigma_{yy} &= \left[\Lambda\left(A^* - I\right) G_{,x} - 2MxG_{,yy}\right] F_x + \\ &\quad + \left[\left(\Lambda + 2M\right)\left(A^* - I\right) G_{,y} - 2MyG_{,yy}\right] F_y; \\ \sigma_{xy} &= \left[A_0 G_{,y} - 2MxG_{,xy}\right] F_x + \left[B_0 G_{,x} - 2MyG_{,xy}\right] F_y; \\ \sigma_{yx} &= \left[B_0 G_{,y} - 2MxG_{,xy}\right] F_x + \left[A_0 G_{,x} - 2MyG_{,xy}\right] F_y. \end{aligned} \tag{23}$$

Here we employ the notation

$$\begin{aligned} A_0 &:= \left(M + \lambda_5 S\right) A^* - M + \lambda_5 S = \left(B - \Lambda\right) A^* - A, \\ B_0 &:= \left(M - \lambda_5 S\right) A^* - M - \lambda_5 S = AA^* - B + \Lambda. \end{aligned}$$

From formula (21) we derive

$$G_{,x} = -\frac{1}{2\pi}\left(\mathrm{I}+\mathrm{A}^*\right)^{-1}\mathrm{A}^{-1}\frac{x}{x^2+y^2}, \quad G_{,y} = -\frac{1}{2\pi}\left(\mathrm{I}+\mathrm{A}^*\right)^{-1}\mathrm{A}^{-1}\frac{y}{x^2+y^2},$$

$$G_{,xy} = -\frac{1}{2\pi}\left(\mathrm{I}+\mathrm{A}^*\right)^{-1}\mathrm{A}^{-1}\frac{2xy}{\left(x^2+y^2\right)^2},$$

$$G_{,xx} = -G_{,yy} = -\frac{1}{2\pi}\left(\mathrm{I}+\mathrm{A}^*\right)^{-1}\mathrm{A}^{-1}\frac{x^2-y^2}{\left(x^2+y^2\right)^2},$$

On the basis of the solution of the Kelvin problem we obtain a solution of the problem for an infinite domain when constant forces $t_x = P_x = (P'_x, P''_x)^T$ and $t_y = P_y = (P'_y, P''_y)^T$ are applied to the interval $\mid x \mid\leq a,\ y = 0$. Let us divide the interval into segments of length $d\xi$. Then the total force applied to an element centered at the point x=ξ, y=0 is equal to $F_\alpha(\xi) = P_\alpha d\xi$, where the subscript α is lither x or y. The solution of the considered problem is obtained if in formulas (22), (23) we introduce the value $F_\alpha(\xi)$, replace x by the expression $x - \xi$ and integrate the solution of (22) and (23) from $-a$ to $+a$. For displacements we obtain

$$\begin{aligned} u_x &= \left(A^*F + yF_{,y}\right)P_x + \left(-yF_{,x}\right)P_y, \\ u_y &= \left(-yF_{,x}\right)P_x + \left(A^*F - yF_{,y}\right)P_y. \end{aligned} \tag{24}$$

Let us introduce the notation

$$S_0 = \left(I+A^*\right)^{-1}A^{-1}, \quad S_1 = \left(\Lambda + 2M\right)A^* - \Lambda, \quad S_2 = 2M - \Lambda\left(A^* - I\right),$$

$$S_3 = \frac{1}{2}\left(A_0 + B_0\right) + 2M, \quad S_4 = \left(\Lambda + 2M\right)\left(A^* - I\right), \quad S_5 = \frac{1}{2}\left(A_0 + B_0\right),$$

$$S_6 = M\left(I + A^*\right), \quad S_7 = B_0 + 2M, \quad S_8 = A_0 + 2M, \quad S_9 = M\left(A^* - I\right).$$

Then stresses are expressed by the formulas

$$\begin{aligned} \sigma_{xx} &= \left(S_1F_{,x} + 2MyF_{,xy}\right)P_x + \left[\Lambda\left(A^* - I\right)F_{,y} + 2MyF_{,yy}\right]P_y, \\ \sigma_{yy} &= \left(-S_2F_{,x} - 2MyF_{,xy}\right)P_x + \left(S_4F_{,y} - 2MyF_{,yy}\right)P_y, \\ \sigma_{xy} &= \left(S_8F_{,y} + 2MyF_{,yy}\right)P_x + \left(B_0F_{,x} - 2MyF_{,xy}\right)P_y, \\ \sigma_{yx} &= \left(S_7F_{,y} + 2MyF_{,yy}\right)P_x + \left(A_0F_{,x} - 2MyF_{,xy}\right)P_y, \end{aligned} \tag{25}$$

where the function *F(x,y)* (2×2 matrix) has the form

$$F(x,y) = \int_{-a}^{a} G(x-\xi,y)\,d\xi = -\tfrac{1}{2\pi} S_0 \left[y \left(\arctan \tfrac{y}{x-a} - \arctan \tfrac{y}{x+a} \right) - \right.$$
$$\left. -(x-a)\ln\sqrt{(x-a)^2+y^2} + (x+a)\ln\sqrt{(x+a)^2+y^2} \right] + C,$$

where $C = (C_1, C_2)^T$, C_1, and C_2 are the constants corresponding to rigid displacement (further they are discarded). For the derivatives *F(x,y)* we have

$$\begin{aligned}
F_{,x} &= \tfrac{1}{2\pi} S_0 \left[\ln\sqrt{(x-a)^2+y^2} - \ln\sqrt{(x+a)^2+y^2} \right], \\
F_{,y} &= -\frac{1}{2\pi} S_0 \left[\arctan\frac{y}{x-a} - \arctan\frac{y}{x+a} \right], \\
F_{,xy} &= \frac{1}{2\pi} S_0 \left[\frac{y}{(x-a)^2+y^2} - \frac{y}{(x+a)^2+y^2} \right], \\
F_{,xx} &= -F_{,yy} = \frac{1}{2\pi} S_0 \left[\frac{x-a}{(x-a)^2+y^2} - \frac{x+a}{(x+a)^2+y^2} \right].
\end{aligned} \tag{26}$$

As seen from formulas (24), displacements u_x and u_y are unbounded at infinity. From formulas (25) it follows that stresses are defined everywhere except for the segment ends ($x = \pm a,\ y = 0$). Their values on the axis $y = 0$ can be obtained if in formulas (25) and (26) y is assumed to be equal to zero

$$\begin{aligned}
\sigma_{xx} &= -\frac{1}{4\pi} S_1 S_0 P_x \ln\left(\frac{x+a}{x-a}\right)^2 - \\
&- \tfrac{1}{2\pi} \Lambda \left(A^* - I\right) S_0 P_y \lim_{y\to\pm 0}\left(\arctan\tfrac{y}{x-a} - \arctan\tfrac{y}{x+a}\right), \\
\sigma_{yy} &= \frac{1}{4\pi} S_2 S_0 P_x \ln\left(\frac{x+a}{x-a}\right)^2 - \\
&- \tfrac{1}{2\pi} S_4 S_0 P_y \lim_{y\to\pm 0}\left(\arctan\tfrac{y}{x-a} - \arctan\tfrac{y}{x+a}\right), \\
\sigma_{xy} &= -\frac{1}{2\pi} S_8 S_0 P_x \lim_{y\to\pm 0}\left(\arctan\frac{y}{x-a} - \arctan\frac{y}{x+a}\right) - \\
&- \tfrac{1}{4\pi} B_0 S_0 P_y \ln\left(\tfrac{x+a}{x-a}\right)^2, \\
\sigma_{yx} &= -\frac{1}{2\pi} S_7 S_0 P_x \lim_{y\to\pm 0}\left(\arctan\frac{y}{x-a} - \arctan\frac{y}{x+a}\right) - \\
&- \tfrac{1}{4\pi} A_0 S_0 P_y \ln\left(\tfrac{x+a}{x-a}\right)^2.
\end{aligned} \tag{27}$$

For the limits contained in formulas (27) the following formula is valid:

$$\lim_{y\to 0}\left(\arctan\frac{y}{x-a} - \arctan\frac{y}{x+a}\right) = \begin{cases} 0, & |x| > a, \quad y = 0_+ \text{ an } y = 0_-, \\ +\pi, & |x| < a, \quad y = 0_+, \\ -\pi & |x| < a, \quad y = 0_-. \end{cases}$$

Taking the latter formula into account, we consider three different cases:

1) $|x| > a, y = 0_\pm$,

$$\sigma_{xx}(x,0) = -\frac{1}{4\pi} S_1 S_0 P_x \ln\left(\frac{x+a}{x-a}\right)^2,$$

$$\sigma_{yy}(x,0) = \tfrac{1}{4\pi} S_2 S_0 P_x \ln\left(\tfrac{x+a}{x-a}\right)^2,$$

$$\sigma_{xy}(x,0) = -\frac{1}{4\pi} B_0 S_0 P_y \ln\left(\frac{x+a}{x-a}\right)^2,$$

$$\sigma_{yx}(x,0) = -\tfrac{1}{4\pi} A_0 S_0 P_y \ln\left(\tfrac{x+a}{x-a}\right)^2.$$

2) $|x| < a, y = 0_+$,

$$\sigma_{xx}(x,0_+) = -\frac{1}{4\pi}S_1S_0P_x\ln\left(\frac{x+a}{x-a}\right)^2 - \frac{1}{2}\Lambda(A^*-I)S_0P_y,$$

$$\sigma_{yy}(x,0_+) = \frac{1}{4\pi}S_2S_0P_x\ln\left(\frac{x+a}{x-a}\right)^2 - \frac{1}{2}S_4S_0P_y, \tag{28}$$

$$\sigma_{xy}(x,0_+) = -\frac{1}{2}S_8S_0P_x - \frac{1}{4\pi}B_0S_0P_y\ln\left(\frac{x+a}{x-a}\right)^2,$$

$$\sigma_{yx}(x,0_+) = -\frac{1}{2}S_7S_0P_x - \frac{1}{4\pi}A_0S_0P_y\ln\left(\frac{x+a}{x-a}\right)^2.$$

3) $|x| < a, y = 0_-$,

$$\sigma_{xx}(x,0_-) = -\frac{1}{4\pi}S_1S_0P_x\ln\left(\frac{x+a}{x-a}\right)^2 + \frac{1}{2}\Lambda(A^*-I)S_0P_y,$$

$$\sigma_{yy}(x,0_-) = \frac{1}{4\pi}S_2S_0P_x\ln\left(\frac{x+a}{x-a}\right)^2 + \frac{1}{2}S_4S_0P_y, \tag{29}$$

$$\sigma_{xy}(x,0_-) = \frac{1}{2}S_8S_0P_x - \frac{1}{4\pi}B_0S_0P_y\ln\left(\frac{x+a}{x-a}\right)^2,$$

$$\sigma_{yx}(x,0_-) = \frac{1}{2}S_7S_0P_x - \frac{1}{4\pi}A_0S_0P_y\ln\left(\frac{x+a}{x-a}\right)^2.$$

As seen from formulas (28) and (29), when being transferred from one side of the segment to the other side, stresses undergo discontinuity

$$\sigma_{xx}(x,0_-) - \sigma_{xx}(x,0_+) = (A^*-I)S_0P_y,$$

$$\sigma_{yy}(x,0_-) - \sigma_{yy}(x,0_+) = S_1S_0P_y,$$

$$\sigma_{xy}(x,0_-) - \sigma_{xy}(x,0_+) = S_2S_0P_x,$$

$$\sigma_{yx}(x,0_-) - \sigma_{yx}(x,0_+) = S_7S_0P_x.$$

3 Coordinate transformation and the influence coefficients

Let the local coordinate system $\overline{x}$, $\overline{y}$ be obtained from the system x, y as a result of its transfer onto (c_x, c_y) and counterclockwise rotation by the angle β .

Using the coordinate transformation formulas [6], displacements and stresses produced by the action of loads $P_{\overline{x}},\ P_{\overline{y}}$ on the segment $|\overline{x}| \leq a,\ \overline{y} = 0$ can be written in terms of the local coordinate system. Displacements have the form

$$u_{\overline{x}} = \left(A^*\overline{F}_1 + \overline{y}\overline{F}_3\right) P_{\overline{x}} + \left(-\overline{y}\overline{F}_2\right) P_{\overline{y}},$$

$$u_{\overline{y}} = \left(-\overline{y}\overline{F}_2\right) P_{\overline{x}} + \left(A^* F_1 - \overline{y}\overline{F}_3\right) P_{\overline{y}},$$

Stresses are expressed by the formulas

$$\sigma_{\overline{x}\overline{x}} = \left[S_1\overline{F}_2 + 2M\overline{y}\overline{F}_4\right] P_{\overline{x}} + \left[\left(S_2 - 2M\right)\overline{F}_3 - 2M\overline{y}\overline{F}_5\right] P_{\overline{y}},$$

$$\sigma_{\overline{y}\overline{y}} = \left[-S_2\overline{F}_2 - 2M\overline{y}\overline{F}_4\right] P_{\overline{x}} + \left[S_4\overline{F}_3 + 2M\overline{y}\overline{F}_5\right] P_{\overline{y}},$$

$$\sigma_{\overline{x}\overline{y}} = \left[S_8\overline{F}_3 - 2M\overline{y}\overline{F}_5\right] P_{\overline{x}} + \left[B_0\overline{F}_2 - 2M\overline{y}\overline{F}_4\right] P_{\overline{y}},$$

$$\sigma_{\overline{y}\overline{x}} = \left[S_7\overline{F}_3 - 2M\overline{y}\overline{F}_5\right] P_{\overline{x}} + \left[A_0\overline{F}_2 - 2M\overline{y}\overline{F}_4\right] P_{\overline{y}}.$$

where

$$F_1(\overline{x},\overline{y}) = F(\overline{x},\overline{y}) = -\frac{1}{2\pi}S_0\left[\overline{y}\left(\arctan\frac{\overline{y}}{\overline{x}-a} - \arctan\frac{\overline{y}}{\overline{x}+a}\right) - \right.$$

$$\left. -(\overline{x}-a)\ln\sqrt{(\overline{x}-a)^2+\overline{y}^2} + (\overline{x}+a)\ln\sqrt{(\overline{x}+a)^2+\overline{y}^2}\right],$$

$$\overline{F}_2(\overline{x},\overline{y}) = F_{,\overline{x}}(\overline{x},\overline{y}) = \frac{1}{2\pi}S_0\left[\ln\sqrt{(\overline{x}-a)^2+\overline{y}^2} - \ln\sqrt{(\overline{x}+a)^2+\overline{y}^2}\right],$$

$$\overline{F}_3(\overline{x},\overline{y}) = F_{,\overline{y}}(\overline{x},\overline{y}) = -\frac{1}{2\pi}S_0\left[\frac{\overline{y}}{(\overline{x}-a)^2+\overline{y}^2} - \frac{\overline{y}}{(\overline{x}+a)^2+\overline{y}^2}\right],$$

$$\overline{F}_4(\overline{x},\overline{y}) = F_{,\overline{xy}}(\overline{x},\overline{y}) = \frac{1}{2\pi}S_0\left[\arctan\frac{\overline{y}}{\overline{x}-a} - \arctan\frac{\overline{y}}{\overline{x}+a}\right],$$

$$\overline{F}_5(\overline{x},\overline{y}) = F_{,\overline{xx}}(\overline{x},\overline{y}) = -F_{,\overline{yy}}(\overline{x},\overline{y}) = \frac{1}{2\pi}S_0\left[\arctan\frac{\overline{y}}{\overline{x}-a} - \right.$$

$$\left. -\arctan\tfrac{\overline{y}}{\overline{x}+a}\right].$$

Using the coordinate transformation formulas [6], the results obtained above can be written in terms of a global coordinate system. Displacements have the form:

$$\begin{aligned} u_x &= \left[A^*\cos\beta\overline{F}_1 + \overline{y}\left(\sin\beta\overline{F}_2 + \cos\beta\overline{F}_3\right)\right]P_{\overline{x}} + \\ &+ \left[-A^*\sin\beta\overline{F}_1 - \overline{y}\left(\cos\beta\overline{F}_2 - \sin\beta\overline{F}_3\right)\right]P_{\overline{y}}, \\ u_y &= \left[A^*\sin\beta\overline{F}_1 - \overline{y}\left(\cos\beta\overline{F}_2 - \sin\beta\overline{F}_3\right)\right]P_{\overline{x}} + \\ &+ \left[A^*\cos\beta\overline{F}_1 - \overline{y}\left(\sin\beta\overline{F}_2 + \cos\beta\overline{F}_3\right)\right]P_{\overline{y}}. \end{aligned} \tag{30}$$

Stresses are written as follows:

$$
\begin{aligned}
\sigma_{xx} = {} & \big[S_1\cos^2\beta\overline{F}_2 - S_2\sin^2\beta\overline{F}_2 - S_3\sin 2\beta\overline{F}_3 + 2M\overline{y}\left(\cos 2\beta\overline{F}_4 + \right.\\
& \left. + \sin 2\beta\overline{F}_5\right)\big] P_{\overline{x}} + \big[(2M - S_2)\cos^2\beta\overline{F}_3 + S_4\sin^2\beta\overline{F}_3\\
& - S_5\sin 2\beta\overline{F}_2 + 2M\overline{y}\left(\sin 2\beta\overline{F}_4 - \cos 2\beta\overline{F}_5\right)\big] P_{\overline{y}},\\
\sigma_{yy} = {} & \big[S_1\sin^2\beta\overline{F}_2 - S_2\cos^2\beta\overline{F}_2 + S_3\sin 2\beta\overline{F}_3 - 2M\overline{y}\left(\cos 2\beta\overline{F}_4 + \right.\\
& \left. + \sin 2\beta\overline{F}_5\right)\big] P_{\overline{x}} + \big[(2M - S_2)\sin^2\beta\overline{F}_3 + S_4\cos^2\beta\overline{F}_3 +\\
& + S_5\sin 2\beta\overline{F}_2 - 2M\overline{y}\left(\sin 2\beta\overline{F}_4 - \cos 2\beta\overline{F}_5\right)\big] P_{\overline{y}},\\
\sigma_{yx} = {} & \big[S_6\sin 2\beta\overline{F}_2 + S_7\cos^2\beta\overline{F}_3 - S_8\sin^2\beta\overline{F}_3 + 2M\overline{y}\left(\sin 2\beta\overline{F}_4\right.\\
& \left. - \cos 2\beta\overline{F}_5\right)\big] P_{\overline{x}} + \big[\left(A_0\cos^2\beta - B_0\sin^2\beta\right)\overline{F}_2 - MS_9\sin 2\beta\overline{F}_3\\
& - 2M\overline{y}\left(\cos 2\beta\overline{F}_4 + \sin 2\beta\overline{F}_5\right)\big] P_{\overline{y}},\\
\sigma_{xy} = {} & \big[S_6\sin 2\beta\overline{F}_2 + S_8\cos^2\beta\overline{F}_3 - S_7\sin^2\beta\overline{F}_3 + 2M\overline{y}\left(\sin 2\beta\overline{F}_4\right.\\
& \left. - \cos 2\beta\overline{F}_5\right)\big] P_{\overline{x}} + \big[\left(B_0\cos^2\beta - A_0\sin^2\beta\right)\overline{F}_2 - S_9\sin 2\beta\overline{F}_3\\
& - 2M\overline{y}\left(\cos 2\beta\overline{F}_4 + \sin 2\beta\overline{F}_5\right)\big] P_{\overline{y}}.
\end{aligned}
\tag{31}
$$

The obtained solutions form the basis for applying one of the boundary element methods called the fictitious load method to the solution of various plane boundary value problems of the binary mixture theory in the case of both finite and infinite domains. To solve these problems the boundary of a given domain is divided sequentially into N segments. If the length of each of these segments is sufficiently small, then we obtain good approximation of the contour. To each boundary element we put into correspondence the concentrated force continuously distributed over this element. For example, to the j-th element there correspond the tangential stress $P_s^j = (P_s^{j\prime}, P_s^{j\prime\prime})^T$ and the normal stress $P_n^j = P_n^{j\prime}, P_n^{j\prime\prime})^T$ continuously distributed over this element. In addition to the fictitious stresses P_s^j and P_n^j, on the j-th element we also consider the true tangential and normal stresses $\sigma_s^j = (\sigma_s^{j\prime}, \sigma_s^{j\prime\prime})^T$ and $\sigma_n^j = (\sigma_n^{j\prime}, \sigma_n^{j\prime\prime})^T$, which are produced by the action of stresses applied to all elements of the boundary.

Using formulas (30), (31) and the coordinate transformation formulas [4], displacements and stresses at the midpoint of the i-th element can be expressed as a function of fictitious loads P_s^j and P_n^j on all N elements of the boundary (i, j = 1,. . .,N). Thus we obtain the equalities

$$\sigma_s^i \equiv \sigma_{r\alpha}^i = \sum_{j=1}^{N_1} \left(A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j \right), \quad \sigma_n^i \equiv \sigma_\alpha^i = \sum_{j=1}^{N_1} \left(A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j \right),$$

$$u_s^i \equiv u_\alpha^i = \textstyle\sum_{j=1}^{N_1} \left(B_{ss}^{ij} P_s^j + B_{sn}^{ij} P_n^j \right), \quad u_n^i \equiv u_r^i = \textstyle\sum_{j=1}^{N_1} \left(B_{ns}^{ij} P_s^j + B_{nn}^{ij} P_n^j \right),$$

$$i = 1, 2, \ldots, N,$$

where $A_{ss}^{ij}, \ldots, B_{nn}^{ij}$ are the boundary influence coefficients. For example, the coefficient A_{ns}^{ij} gives the normal stress σ_{ni} at the center of the i-th segment produced by the action of the unit tangential load on the j-th segment.

Thus the problem is reduced to finding the fictitious loads P_s^j and P_n^j using the above boundary conditions, i.e. to the solution of a system of linear algebraic equations, where P_s^j and P_n^j are the unknowns. Having solved this system and using formulas (30), (31) and the coordinate transformation formulas at an arbitrary point of the considered domain we obtain the values of the displacement vector and stress tensor components.

As an illustration, in the next section we give examples of the numerical realization of some boundary value problems for a binary mixture and also present the corresponding diagrams.

5. Examples

Below, using the boundary element method (BEM), we give solutions of two static boundary value problems for an elastic body consisting of a binary mixture. The first of them is an external problem for an infinite domain with a circular hole when the contour is stress-free and unilateral shearing stresses is applied at infinity. The second problem concerns a circular semi-ring when stresses are given at two opposite semi-circles and the symmetry and antisymmetry conditions are given on two opposite interval segments [7].

Problem 1 We consider a boundary value problem, in the domain

$\Omega = \{r_1 < r < \infty,\ 0 < \alpha < 2\pi\}$ with the following boundary conditions::

$$\text{when} \quad r = r_1\colon\ \sigma_{rr} = (0,0)^T, \quad \sigma_{r\alpha} = (0,0)^T,$$

$$\text{when} \quad r \to \infty\colon\ \sigma_{xx} = p = (p', p'')^T, \quad \sigma_{yy} = \sigma_{xy} = \sigma_{yx} = (0,0)^T.$$

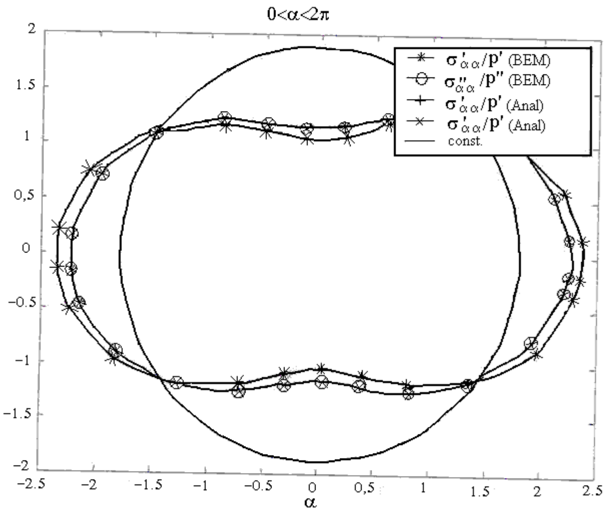

Figure 1: Tangential stresses on the hole boundary

Since the problem has two symmetry axes, the numerical solution can be found by dividing a quarter of the circular boundary into 50 elements; in this case $\lambda_1 = 0,1$; $\lambda_2 = 0,2$; $\lambda_3 = 0,3$; $\lambda_4 = 0,4$; $\lambda_5 = 0,5$; $\mu_1 = 0,6$; $\mu_2 = 0,7$; $\mu_3 = 0,8$; $\rho_1 = 0,15$; $\rho_2 = 0,25$; $p'/E' = 10^-3$; $p''/E'' = 15.10^-4$, $r_1 = 1.75$, $0 < \alpha < 2\pi$.

An analytic solution for stresses acting along the circular hole boundary has the form [8]

$$\text{when } r = r_1: \quad \sigma_{\alpha\alpha} = \left\{ \mathrm{I} - \left[\mathrm{I} + \mathrm{M}\,(\mathrm{A} - \lambda_5 S \mathrm{A}^*)^{-1} \right] \cos 2\alpha \right\} p,$$
$$\sigma_{\alpha r} = -\left\{ \left[\mathrm{I} - \mathrm{M}\,(\mathrm{A} - \lambda_5 S \mathrm{A}^*)^{-1} \right] \sin 2\alpha \right\} p,$$

where the angle α is counted from the x-axis. These functions are shown in Fig.1 together with numerical results.

The comparison of the results obtained by the boundary element method with the exact solution values shows a high degree of their coincidence (see Fig.1). We can therefore conclude that the application of the BEM has proved to be correct for solving the boundary value problems considered in this paper.

Problem 2 Now let us consider the boundary value problem considered

in the domain $\Omega = \{r_1 < r < r_2, 0 < \alpha < \pi\}$ with the following boundary conditions:

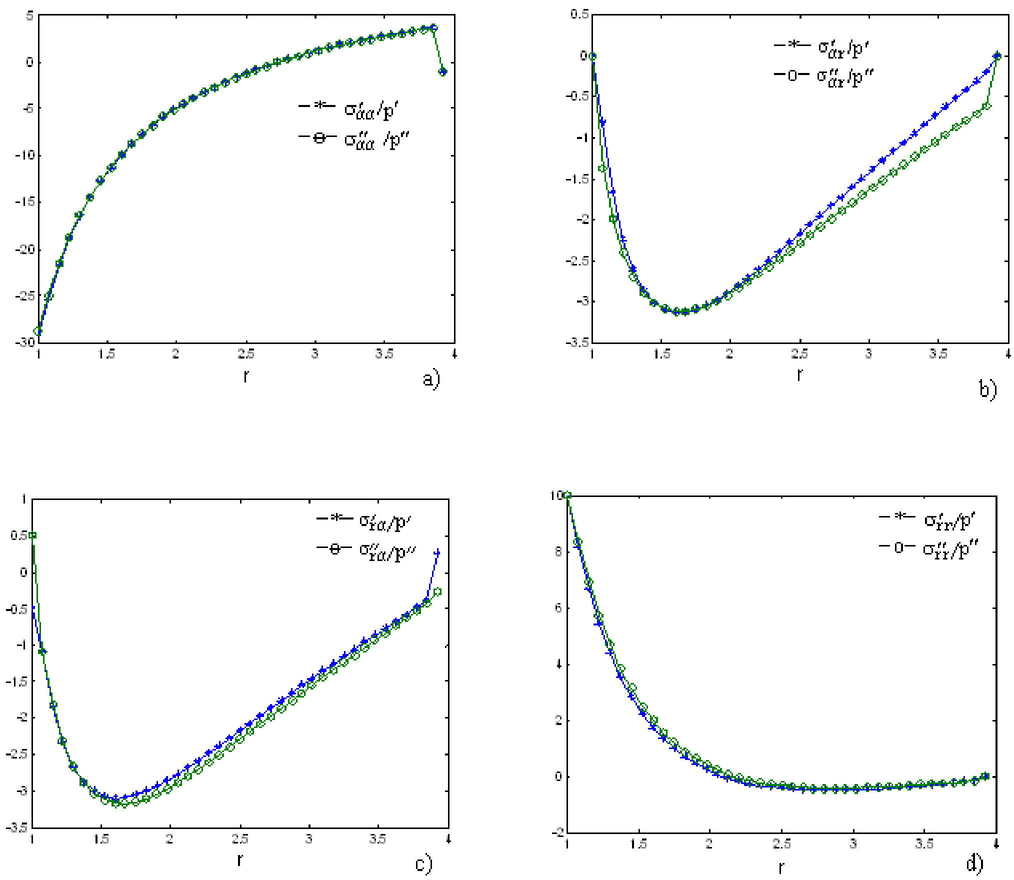

Figure 2: a) Tangential, b), c) shearing and d) normal stresses in the ring $(1 < r < 4)$ for $\alpha = \pi/3$

$$
\begin{aligned}
&a)\quad \text{when}\ \ r = r_1:\quad \sigma_{rr} = \left(p' \cos\tfrac{\alpha}{2}, p'' \cos\tfrac{\alpha}{2}\right)^T, \quad \sigma_{r\alpha} = (0,0)^T,\\
&b)\quad \text{when}\ \ r = r_2:\quad \sigma_{rr} = (0,0)^T, \quad \sigma_{r\alpha} = (0,0)^T,\\
&c)\quad \text{when}\ \ \alpha = 0:\quad v = (0,0)^T, \quad\ \ \sigma_{r\alpha} = (0,0)^T,\\
&d)\quad \text{when}\ \ \alpha = \pi.:\quad u = (0,0)^T, \quad \sigma_{\alpha\alpha} = (0,0)^T.
\end{aligned}
$$

The formulated problem is solved by the method of boundary elements. At the characteristic points of the considered domain we have obtained the stress values for the following data: $\lambda_1 = 0.1$; $\lambda_2 = 0,2$; $\lambda_3 = 0.3$; $\lambda_4 = 0.4$; $\lambda_5 = 0.5$; $\mu_1 = 0.6$; $\mu_2 = 0.7$; $\mu_3 = 0.8$; $\rho_1 = 0.15$; $\rho_2 = 0.25$; $p'/E' = 10^-3$; $p''/E'' = 10^-3$; $r_1 = 1$; $r_2 = 4$. The semicircles $r = r_1$ and $r = r_2$ are divided into 180 equal arcs, while the linear parts of the boundary are divided into 40 equal segments. The diagrams for the stresses $\sigma_{\alpha\alpha}/p = (\sigma'_{\alpha\alpha}/p', \sigma''_{\alpha\alpha}/p'')^T$, $\sigma_{\alpha r}/p = (\sigma'_{\alpha r}/p', \sigma''_{\alpha r}/p'')^T$, $\sigma_{r\alpha}/p = (\sigma'_{r\alpha}/p', \sigma''_{r\alpha}/p'')^T$, $\sigma_{rr}/p = (\sigma'_{rr}/p', \sigma''_{rr}/p'')^T$ are shown in Fig.2 for $r_1 < r < r_2$, $\alpha = \pi/3$.

By solving the problems corresponding to Fig. 2 we obtain the picture of distribution of internal stresses throughout the body. In particular, using the BEM we have calculated the distribution of stresses $\sigma_{\alpha\alpha}$, $\sigma_{r\alpha}$, $\sigma_{\alpha,r}$, σ_{rr} along the radius r when $\alpha = \dfrac{\pi}{3}$.

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