THE CONTACT PROBLEM OF STATICS FOR A THERMOELASTIC MIXTURE

I. Tsagareli

I.Vekua Institute of Applied Mathematics
Tbilisi State University

(Received: 01.03.09; accepted: 12.07.09)

Abstract

In the present paper we consider the contact problem for a piecewise-homogeneous plane consisting of two domains filled with different binary elastic mixtures. On the interface there are prescribed: the difference of limiting vector values of partial displacements for each domain; difference of limiting vector values of partial thermal stresses; difference of limiting values of temperature changes and difference of heat flows. The solution is given in the form of absolutely and uniformly convergent series which allow one to perform numerical analysis of the problem. The question on the uniqueness of the solution of the problem is studied.

Key words and phrases: Binary mixtures for thermoelastic solids, contact problems, theorems of uniqueness.

AMS subject classification: 74E30,74M15, 74G10, 74G30.

Let the circle S, of radius R, divide a plane into inner and outer domains which are denoted by D_0 and D_1, respectively. Every field is filled with different elastic mixtures.

We formulate the following problem: Find in the field D_j $(j = 0, 1)$ a regular solution $\overset{j}{U}(x) = (\overset{j}{u}(x), \overset{j}{u}_3(x))$, $\overset{j}{U}(x) \in C^2(D_j) \cap C^1(\overline{D}_j)$ $(\overline{D}_j = D_j \cup S)$ of a system of equations of statics of the theory of thermoelastic mixtures [1-3]:

$$\overset{j}{a}_1 \Delta \overset{j}{u}{}^1(x) + \overset{j}{b}_1 \operatorname{grad}\operatorname{div} \overset{j}{u}{}^1(x) + \overset{j}{c}\Delta \overset{j}{u}{}^2(x) + \overset{j}{d}\operatorname{grad}\operatorname{div}\overset{j}{u}{}^2(x) = \overset{j}{\gamma}_1 \operatorname{grad} \overset{j}{u}_3(x),$$

$$\overset{j}{c}\Delta \overset{j}{u}{}^1(x) + \overset{j}{d}\operatorname{grad}\operatorname{div} \overset{j}{u}{}^1(x) + \overset{j}{a}_2\Delta \overset{j}{u}{}^2(x) + \overset{j}{b}_2\operatorname{grad}\operatorname{div}\overset{j}{u}{}^2(x) = \overset{j}{\gamma}_2 \operatorname{grad} \overset{j}{u}_3(x),$$

$$\Delta \overset{j}{u}_3(x) = 0, \quad j = 0, 1, \tag{1}$$

such that on the circumference S it satisfies the contact conditions

$$\overset{1}{u}{}^-(z) - \overset{0}{u}{}^+(z) = f(z), \quad z \in S,$$

$$[\overset{1}{\mathbb{R}}(\partial_z, n)\overset{1}{U}(z)]^- - [\overset{0}{\mathbb{R}}(\partial_z, n)\overset{0}{U}(z)]^+ = F(z), \tag{2}$$

$$\overset{1}{u}_3{}^-(z) - \overset{0}{u}_3{}^+(z) = f_3(z), \quad \alpha\Big[\frac{d\overset{1}{u}_3(z)}{dn(z)}\Big]^- - \beta\Big[\frac{d\overset{0}{u}_3(z)}{dn(z)}\Big]^+ = f_4(z) \qquad (3)$$

and the conditions

$$\overset{1}{U}(x) = O(1), \quad r^2\frac{\partial\overset{1}{U}(x)}{\partial_{x_k}} = O(1), \quad k = 1,2; \quad x = (x_1, x_2), \quad x \in D_1$$

at infinity. Here $\overset{j}{u}(x) = (\overset{j}{u}{}^1(x), \overset{j}{u}{}^2(x))$ and $\overset{j}{u}{}^i(x) = (\overset{j}{u}_1{}^i(x), \overset{j}{u}_2{}^i(x))$ are partial displacements in D_j; $\overset{j}{u}_3(x)$ is temperature change in D_j, $j = 0,1$; $i = 1,2$.

$$\overset{j}{\mathbb{R}}(\partial_x, n)\overset{j}{U}(x) = \Big[(\overset{j}{\mathbb{R}}(\partial_x, n)\overset{j}{U}(x))^1, (\overset{j}{\mathbb{R}}(\partial_x, n)\overset{j}{U}(x))^2\Big],$$

$$(\overset{j}{\mathbb{R}}(\partial_x, n)\overset{j}{U}(x))^i = \Big[(\overset{j}{\mathbb{R}}(\partial_x, n)\overset{j}{U}(x))_1{}^i, (\overset{j}{\mathbb{R}}(\partial_x, n)\overset{j}{U}(x))_2{}^i\Big],$$

are partial thermal stresses of the mixture in D_j,

$$(\overset{j}{\mathbb{R}}(\partial_x, n)\overset{j}{U}(x))^i_p = (\overset{j}{\mathbb{P}}(\partial_x, n)\overset{j}{u}(x))^i_p - \overset{j}{\gamma}_i n_p(x)\overset{j}{u}_3(x), \quad p = 1,2; \qquad (4)$$

$\overset{j}{\mathbb{P}}(\partial_x, n)\overset{j}{u}(x)$ is the stress vector [3]:

$$(\overset{j}{\mathbb{P}}(\partial_x, n)\overset{j}{u}(x))^1_p = \sum_{q=1}^{2}[(\overset{j}{\lambda}_1\overset{j}{\theta}_1(x) + \overset{j}{\lambda}_3\overset{j}{\theta}_2(x))\delta_{pq} + 2\overset{j}{\mu}_1\overset{j}{\varepsilon}{}^1_{qp}(x) +$$

$$+2\overset{j}{\mu}_3\overset{j}{\varepsilon}{}^2_{qp}(x) - 2\overset{j}{\lambda}_5\overset{j}{h}_{qp}(x)]n_q - \overset{j}{\rho}{}^{-1}\overset{j}{\alpha}_2(\overset{j}{\rho}_2\overset{j}{\theta}_1(x) + \overset{j}{\rho}_1\overset{j}{\theta}_2(x))n_p;$$

$$(\overset{j}{\mathbb{P}}(\partial_x, n)\overset{j}{u}(x))^2_p = \sum_{q=1}^{2}[(\overset{j}{\lambda}_4\overset{j}{\theta}_1(x) + \overset{j}{\lambda}_2\overset{j}{\theta}_2(x))\delta_{pq} + 2\overset{j}{\mu}_3\overset{j}{\varepsilon}{}^1_{qp}(x) +$$

$$+2\overset{j}{\mu}_2\overset{j}{\varepsilon}{}^2_{qp}(x) + 2\overset{j}{\lambda}_5\overset{j}{h}_{qp}(x)]n_q + \overset{j}{\rho}{}^{-1}\overset{j}{\alpha}_2(\overset{j}{\rho}_2\overset{j}{\theta}_1(x) + \overset{j}{\rho}_1\overset{j}{\theta}_2(x))n_p,$$

$$\overset{j}{\varepsilon}{}^i_{qp}(x) = \frac{1}{2}(\partial_q\overset{j}{u}{}^i_p(x) + \partial_p\overset{j}{u}{}^i_q(x)), \quad \overset{j}{\theta}_p(x) = \operatorname{div}\overset{j}{u}{}^p(x),$$

$$\overset{j}{h}_{qp} = \frac{1}{2}[\partial_q(\overset{j}{u}{}^1_p - \overset{j}{u}{}^2_p) - \partial_p(\overset{j}{u}{}^1_q - \overset{j}{u}{}^2_q)],$$

$$\partial_p = \frac{\partial}{\partial_{x_p}}, \quad i,p,q = 1,2; \quad j = 0,1,$$

δ_{pq} is the Kronecker symbol, $f(z) = (f^1(z), f^2(z)) = (f^1_1, f^1_2, f^2_1, f^2_2)$; $F(z) = (F^1(z), F^2(z)) = (F^1_1, F^1_2, F^2_1, F^2_2)$, $f_3(z)$, $f_4(z)$ are the known on S functions,

$$\int_S F(y)d_yS = 0, \quad \int_S f_4(y)d_yS = 0, \quad \int_S [y\times F(y)]d_yS = 0, \quad y = (y_1, y_2) \in S,$$

$r^2 = x_1^2 + x_2^2$; $\overset{j}{a}_1 = \overset{j}{\mu}_1 - \overset{j}{\lambda}_5$, $\overset{j}{b}_1 = \overset{j}{\mu}_1 + \overset{j}{\lambda}_5 + \overset{j}{\lambda}_1 - \overset{j}{\rho}{}^{-1}\overset{j}{\rho}_2\overset{j}{\alpha}_2$, $\overset{j}{a}_2 = \overset{j}{\mu}_2 - \overset{j}{\lambda}_5$, $\overset{j}{b}_2 = \overset{j}{\mu}_2 + \overset{j}{\lambda}_2 + \overset{j}{\lambda}_5 + \overset{j}{\rho}{}^{-1}\overset{j}{\rho}_2\overset{j}{\alpha}_2$, $\overset{j}{c} = \overset{j}{\mu}_3 + \overset{j}{\lambda}_5$, $\overset{j}{d} = \overset{j}{\mu}_3 + \overset{j}{\lambda}_3 - \overset{j}{\lambda}_5 - \overset{j}{\rho}{}^{-1}\overset{j}{\rho}_1\overset{j}{\alpha}_2 \equiv \overset{j}{\mu}_3 + \overset{j}{\lambda}_4 - \overset{j}{\lambda}_5 + \overset{j}{\rho}{}^{-1}\overset{j}{\rho}_2\overset{j}{\alpha}_2$, $\overset{j}{\alpha}_2 = \overset{j}{\lambda}_3 - \overset{j}{\lambda}_4$; $\overset{j}{\lambda}_1$, $\overset{j}{\lambda}_2$, $\overset{j}{\lambda}_3$, $\overset{j}{\lambda}_4$, $\overset{j}{\lambda}_5$, $\overset{j}{\mu}_1$, $\overset{j}{\mu}_2$, $\overset{j}{\mu}_3$, $\overset{j}{\gamma}_1$, $\overset{j}{\gamma}_2$ are the constants characterizing elastic and thermal properties of the mixture in D_j, $j = 0, 1$; α and β are the positive constants.

In a static case, the problems [(1)$_3$,(3)] regarding temperature and [(1), (2)] regarding displacements can be solved separately. The question of the uniqueness of solutions of these problems can likewise be studied separately.

The following theorems are valid.

Theorem 1. *A regular solution of the problem* $[(1), (2)]$ *is defined modulo an arbitrary constant vector.* **Theorem 2.** *A regular solution of the problem* $[(1)_3, (3)]$ *is defined modulo an arbitrary constant value.*

Indeed, let the problem [(1), (2)] have two regular solutions. The difference of these solutions, obviously, satisfies the homogeneous system $(1)_0$ and zero contact conditions $(2)_0$. To find the difference in each of the domains D_1 and D_2, we use Green's formula. Moreover, taking into account the identity

$$\overset{0}{u}{}^{+}(z)\Big[\overset{0}{\mathbb{P}}(\partial_z, n)\overset{0}{u}(z)\Big]^{+} - \overset{1}{u}{}^{-}(z)\Big[\overset{1}{\mathbb{P}}(\partial_z, n)\overset{1}{u}(z)\Big]^{-} =$$
$$= (\overset{0}{u}{}^{+}(z) - \overset{1}{u}{}^{-}(z))\Big[\overset{0}{\mathbb{P}}(\partial_z, n)\overset{0}{u}(z)\Big]^{+} +$$
$$+\overset{1}{u}{}^{-}(z)\Big\{\Big[\overset{0}{\mathbb{P}}(\partial_z, n)\overset{0}{u}(z)\Big]^{+} - \Big[\overset{1}{\mathbb{P}}(\partial_z, n)\overset{1}{u}(z)\Big]^{-}\Big\},$$

from the homogeneous contact conditions $(2)_0$, as well as from the conditions at infinity we can state that the difference of any two regular soutions of the problem is a constant vector.

Using now for the domains D_0 and D_1 Green's (Dirichlet) formulas for the Laplace equation, from the homogeneous conditions $(3)_0$ and by means of the identities

$$\overset{0}{u}_3{}^{+}(z)\beta\Big[\frac{d\overset{0}{u}_3(z)}{dn(z)}\Big]^{+} - \overset{1}{u}_3{}^{-}(z)\alpha\Big[\frac{d\overset{1}{u}_3(z)}{dn(z)}\Big]^{-} =$$
$$= (\overset{0}{u}_3{}^{+}(z) - \overset{1}{u}_3{}^{-}(z))\beta\Big[\frac{d\overset{0}{u}_3(z)}{dn(z)}\Big]^{+} + \overset{1}{u}_3{}^{-}(z)\Big\{\beta\Big[\frac{d\overset{0}{u}_3(z)}{dn(z)}\Big]^{+} - \alpha\Big[\frac{d\overset{1}{u}_3(z)}{dn(z)}\Big]^{-}\Big\},$$

we obtain

$$\frac{\partial\overset{j}{u}_3(x)}{\partial_{x_i}} = 0, \quad j = 1, 2; \quad j = 0, 1.$$

Which allows us to conclude that the difference of two solutions of the problem $[(1)_3,(3)]$ is an arbitrary constant $\overset{j}{u}_3(x)=\text{const}$.

On the basis of the above theorems, we can now formulate the uniqueness theorem of the above-posed problem.

Theorem 3. *A regular solution of the contact problem of thermoelastic mixture under consideration is defined modulo an arbitrary constant vector.*

We will now proceed to solving the problem $[(1)_3,(3)]$. A harmonic function $\overset{j}{u}_3(x)$ is sought in the domain D_j in the form of a series

$$\overset{j}{u}_3(x)=\sum_{m=0}^{\infty}\overset{j}{f}_{3m}(x),\quad j=0,1, \tag{5}$$

where

$$\overset{0}{f}_{3m}(x)=\Big(\frac{r}{R}\Big)^m(\overset{0}{X}_m\cdot\nu_m(\psi)),\quad \overset{1}{f}_{3m}(x)=\Big(\frac{R}{r}\Big)^m(\overset{1}{X}_m\cdot\nu_m(\psi)), \tag{6}$$

$\overset{j}{X}_m$ are the unknown two-component constant vectors, $\nu_m(\psi)=(\cos m\psi,\sin m\psi)$.

Let the functions $f_3(z)$ and $f_4(z)$ be expanded into the Fourier series. Substituting formulas (5) and (6) into the conditions (3) and passing to the limit, as $r\to R$, we obtain the system of equations with respect to $\overset{j}{X}_m$, $j=0,1$. Solving this system, we find values of the unknown quantities:

$$\overset{0}{X}_m=-\frac{\beta_m+\alpha m\alpha_m}{(\alpha+\beta)m},\quad \overset{1}{X}_m=-\frac{\beta_m-\beta m\alpha_m}{(\alpha+\beta)m},\quad m=1,2,\dots. \tag{7}$$

Taking into account the condition $\int\limits_S f_4(y)dyS=0$ for $m=0$, we obtain $\overset{1}{X}_0=\overset{0}{X}_0+\alpha_0$, where $\alpha_m=(\alpha_{m1},\alpha_{m2})$ and $\beta_m=(\beta_{m1},\beta m2)$ are the Fourier coefficients of the functions $f_3(z)$ and $f_4(z)$, $\alpha_0=(\alpha_{01},0)$ respectively, $\alpha_{01}=\frac{1}{\pi}\int\limits_0^{2\pi}f_3(\theta)d\theta$, $x=(r,\psi)$, $z=(R,\psi)$, $y=(R,\theta)$.

Let us now solve the problem $[(1),(2)]$. The condition (2) with regard for (4) can be replaced by the condition

$$\begin{aligned}&\Big\{\Big[\overset{1}{\mathbb{P}}(\partial_z,n)\overset{1}{u}(z)\Big]^i\Big\}^- -\Big\{\Big[\overset{0}{\mathbb{P}}(\partial_z,n)\overset{0}{u}(z)\Big]^i\Big\}^+=\\&=F^i(z)+n(z)\Big[\overset{1}{\gamma}_i\overset{1}{u}_3^-(z)-\overset{0}{\gamma}_i\overset{0}{u}_3^+(z)\Big],\end{aligned} \tag{8}$$

where $\overset{0}{u}_3^{+}$ and $\overset{1}{u}_3^{-}$ are defined by formulas (5), (6) and (7). Thus we pass now to the problem [(1), (8)]. A solution of this problem is sought in the form of a sum

$$\overset{j}{u}(x) = \overset{j}{w}_1(x) + \overset{j}{w}_2(x), \quad j = 0, 1, \tag{9}$$

where $\overset{j}{w}_1(x)$ is the solution of the homogeneous system $(1)_0$ with the conditions

$$\begin{gathered} \Big\{\Big[\overset{1}{w}_1(z)\Big]^i\Big\}^- - \Big\{\Big[\overset{0}{w}_1(z)\Big]^i\Big\}^+ = f^i(z), \\ \Big\{\Big[\overset{1}{\mathbb{P}}(\partial_z, n)\overset{1}{w}_1(z)\Big]^i\Big\}^- - \Big\{\Big[\overset{0}{\mathbb{P}}(\partial_z, n)\overset{0}{w}_1(z)\Big]^i\Big\}^+ = F^i(z), \end{gathered} \tag{10}$$

and $\overset{j}{w}_2(x)$ is the solution of the system (1) with the

$$\begin{gathered} \Big\{\Big[\overset{1}{w}_2(z)\Big]^i\Big\}^- - \Big\{\Big[\overset{0}{w}_2(z)\Big]^i\Big\}^+ = 0, \\ \Big\{\Big[\overset{1}{\mathbb{P}}(\partial_z, n)\overset{1}{w}_2(z)\Big]^i\Big\}^- - \Big\{\Big[\overset{0}{\mathbb{P}}(\partial_z, n)\overset{0}{w}_2(z)\Big]^i\Big\}^+ = n\Big[\overset{1}{\gamma}_i\overset{1}{u}_3 1^-(z) - \overset{0}{\gamma}_i\overset{0}{u}_3 0^+(z)\Big]. \end{gathered} \tag{11}$$

To find a solution $\overset{j}{w}_1(x)$ of the problem $[(1)_0, (10)]$, we use a general representation of solutions of the system $(1)_0$ in the plane, which is expressed by four harmonic functions $\overset{j}{\Phi}_k(x)$ [4]:

$$\begin{gathered} \overset{j}{w}{}_1^{1}(x) = \operatorname{grad}\overset{j}{\Phi}_1(x) + \\ + r^2 \operatorname{grad}\Big\{\Big[(\overset{j}{\xi}_1 + \frac{1}{2})r\frac{\partial}{\partial r} + 2\overset{j}{\xi}_1\Big]\overset{j}{\Phi}_2(x) + \overset{j}{\beta}_1(r\frac{\partial}{\partial r} + 2)\overset{j}{\Phi}_3(x)\Big\} - \\ - xr\frac{\partial}{\partial r}[(2\overset{j}{\xi}_1 - 1) - \overset{j}{\Phi}_2(x) + 2\overset{j}{\beta}_1\overset{j}{\Phi}_3(x) + \overset{j}{\Psi}{}^1(x), \\ \overset{j}{w}{}_1^{2}(x) = \operatorname{grad}\overset{j}{\Phi}_4(x) + r^2\operatorname{grad}\Big\{(\overset{j}{\xi}_2(r\frac{\partial}{\partial r} + 2)\overset{j}{\Phi}_2(x) + \\ + \Big[(\overset{j}{\beta}_2 + \frac{1}{2})r\frac{\partial}{\partial r} + 2\overset{j}{\beta}_2\Big]\overset{j}{\Phi}_3(x)\Big\} - \\ - xr\frac{\partial}{\partial r}\Big[2\overset{j}{\xi}_2(x)\overset{j}{\Phi}_2(x) + (2\overset{j}{\beta}_2 - 1)\overset{j}{\Phi}_3(x)\Big] + \overset{j}{\Psi}{}^2(x), \end{gathered} \tag{12}$$

where

$$\overset{0}{\Psi}{}^i(x) = \overset{0}{A}{}^i x + \overset{0}{B}{}^i\widetilde{x}, \quad x \in D_0; \quad \overset{1}{\Psi}{}^i(x) = \frac{1}{r^2}\Big[\overset{1}{A}{}^i x + \overset{1}{B}{}^i\widetilde{x}\Big], \quad x \in D_1,$$

$$\overset{j}{\xi}_1 = \frac{1}{2\overset{j}{\Delta}_1}(\overset{j}{c}\overset{j}{d} - \overset{j}{b}_1\overset{j}{a}_2 - \overset{j}{\Delta}_1), \quad \overset{j}{\beta}_1 = \frac{1}{2\overset{j}{\Delta}_1}(\overset{j}{c}\overset{j}{b}_2 - \overset{j}{a}_2\overset{j}{d}),$$

$$\overset{j}{\beta}_2 = \frac{1}{2\overset{j}{\Delta}_1}(\overset{j}{c}\overset{j}{d} - \overset{j}{b}_1\overset{j}{a}_2 - \overset{j}{\Delta}_1), \quad \overset{j}{\xi}_2 = \frac{1}{2\overset{j}{\Delta}_1}(\overset{j}{c}\overset{j}{b}_1 - \overset{j}{a}_1\overset{j}{d}), \quad \overset{j}{\Delta}_1 = \overset{j}{a}_1\overset{j}{a}_2 - (\overset{j}{c})^2,$$
$$x = (x_1, x_2), \quad \widetilde{x} = (-x_2, x_1),$$

$\overset{j}{A}i^i$ and $\overset{j}{B}i^i$ are the unknown constants, $i = 1, 2, \quad j = 0, 1$.

The functions $\overset{j}{\Phi}_k(x)$ are sought in the form of the following series:

$$\overset{0}{\Phi}_k(x) = \sum_{m=0}^{\infty}\left(\frac{r}{R}\right)^m(\overset{0}{X}_{mk}\cdot\nu_m(\psi)), \quad x \in D_0,$$
$$\overset{1}{\Phi}_k(x) = \sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^m(\overset{1}{X}_{mk}\cdot\nu_m(\psi)), \quad x \in D_1, \quad k = 1, 2, 3, 4, \tag{13}$$

where $\overset{j}{X}_{mk}$ is the unknown two-component constant vector, $\nu_m(\psi) = (\cos m\psi, \sin m\psi)$, $x = (r, \psi)$.

We substitute (13) into (12) and the above-obtained expression into (10). The functions $\overset{j}{f}(z)$ and $\overset{j}{F}(z)$ are assumed to be reprezentable by Fourier series. Passing to the limit, as $r \to R$, for every value m, from (10) we obtain with regard to the unknowns $\overset{0}{X}_{mk}$ and $\overset{1}{X}_{mk}$ the system of linear algebraic equations:

$$\sum_{p=1}^{4}\overset{j}{a}_{kp}\overset{j}{X}_{mp} = \overset{j}{\chi}_{mk}, \quad k = 1, 2, 3, 4; \quad j = 0, 1; \quad m = 1, 2, \dots, \tag{14}$$

where for the case of $j = 0$, i.e. for the domain D_0, the following notation are introduced:

$$\overset{0}{\chi}_{m1} = \frac{\overset{0}{\zeta}_m{}^1}{m}, \quad \overset{0}{\chi}_{m2} = \frac{\overset{0}{\eta}_m{}^1}{m}, \quad \overset{0}{\chi}_{m3} = \frac{\overset{0}{\zeta}_m{}^2}{m}, \quad \overset{0}{\chi}_{m4} = \frac{\overset{0}{\eta}_m{}^2}{m};$$
$$\overset{0}{a}_{11} = \frac{1}{R^2}(2\overset{0}{\mu}_1 m - \varepsilon_1), \quad \overset{0}{a}_{12} = 2\left[\overset{0}{\mu}_1(\overset{0}{\xi}_1 + \frac{1}{2}) + \overset{0}{\xi}_2\overset{0}{\mu}_3\right]m^2 +$$
$$+(\overset{0}{\xi}_2\varepsilon_2 - 2\overset{0}{\xi}_1\varepsilon_4 + \varepsilon_1(\overset{0}{\xi}_1 + \frac{3}{2}) - 2\overset{0}{\xi}_2\varepsilon_5)m + \varepsilon_1; \ \overset{0}{a}_{13} = 2\left[\overset{0}{\mu}_1\overset{0}{\beta}_1 + (\overset{0}{\beta}_2 + \frac{1}{2})\overset{0}{\mu}_3\right]m^2 +$$
$$+\left[\varepsilon_2(\overset{0}{\beta}_2 + \frac{3}{2}) + \varepsilon_1\overset{0}{\beta}_1 - 2\overset{0}{\beta}_2\varepsilon_5 - 2\overset{0}{\beta}_1\varepsilon_4\right]m + \varepsilon_2;$$
$$\overset{0}{a}_{14} = \frac{1}{R^2}(2\overset{0}{\mu}_3 m - \varepsilon_2); \ \overset{0}{a}_{21} = \frac{1}{R^2}(2\overset{0}{\lambda}_5 m - \varepsilon_6); \ \overset{0}{a}_{22} = 2\left[\overset{0}{\lambda}_5(\overset{0}{\xi}_1 + \frac{1}{2}) - \overset{0}{\xi}_2\overset{0}{\lambda}_5\right]m^2 +$$
$$+(3\overset{0}{\xi}_1\varepsilon_6 + 3\overset{0}{\xi}_2\varepsilon_7 + \frac{1}{2}\overset{0}{\xi}_6 - \overset{0}{a}_1)m + 2\varepsilon_6\overset{0}{\xi}_1 + 2\overset{0}{\xi}_2\varepsilon_7;$$

$$\overset{0}{a}_{23} = 2\Big[\overset{0}{\lambda}_5\overset{0}{\beta}_1 - \overset{0}{\lambda}_5(\overset{0}{\beta}_2 + \frac{1}{2})\Big]m^2 + (3\varepsilon_6\overset{0}{\beta}_1 + 3\varepsilon_7\overset{0}{\beta}_2 + \frac{1}{2}\varepsilon_7 - \overset{0}{c})m+$$
$$+2\varepsilon_6\overset{0}{\beta}_1 + 2\varepsilon_7\overset{0}{\beta}_2; \quad \overset{0}{a}_{24} = -\frac{1}{R^2}(2\overset{0}{\lambda}_5 m + \varepsilon_7);$$
$$\overset{0}{a}_{31} = \frac{1}{R^2}(2\overset{0}{\mu}_3 m - \varepsilon_2); \quad \overset{0}{a}_{32} = 2\Big[\overset{0}{\mu}_3(\overset{0}{\xi}_1 + \frac{1}{2}) + \overset{0}{\xi}_2\overset{0}{\mu}_2\Big]m^2+$$
$$+\Big[\varepsilon_2(\overset{0}{\xi}_1+\frac{3}{2})+\varepsilon_3\overset{0}{\xi}_2-2\varepsilon_5\overset{0}{\xi}_4-2\varepsilon_9\overset{0}{\xi}_2\Big]m+\varepsilon_2; \; \overset{0}{a}_{33} = 2\Big[\overset{0}{\mu}_3\overset{0}{\beta}_1 + \overset{0}{\mu}_2(\overset{0}{\beta}_2 + \frac{1}{2})\Big]m^2+$$
$$+\Big[\varepsilon_3(\overset{0}{\beta}_2 + \frac{3}{2}) + \varepsilon_2\overset{0}{\beta}_1 - 2\varepsilon_5\overset{0}{\beta}_1 - 2\varepsilon_9\overset{0}{\beta}_2\Big]m + \varepsilon_3; \quad \overset{0}{a}_{34} = \frac{1}{R^2}(2\overset{0}{\mu}_2 m - \varepsilon_3);$$
$$\overset{0}{a}_{41} = -\frac{1}{R}(2\overset{0}{\lambda}_5 m + \varepsilon_7); \quad \overset{0}{a}_{42} = -2\Big[\overset{0}{\lambda}_5(\overset{0}{\xi}_1 + \frac{1}{2}) - \overset{0}{\lambda}_5\overset{0}{\xi}_2\Big]m^2+$$
$$+(3\varepsilon_7\overset{0}{\xi}_1 + 3\varepsilon_8\overset{0}{\xi}_2 + \frac{1}{2}\varepsilon_7 - \overset{0}{c})m + 2\varepsilon_7\overset{0}{\xi}_1 + 2\varepsilon_8\overset{0}{\xi}_2;$$
$$\overset{0}{a}_{43} = -2\Big[\overset{0}{\lambda}_5\overset{0}{\beta}_1 - \overset{0}{\lambda}_5(\overset{0}{\beta}_2 + \frac{1}{2})\Big]m^2 + (3\overset{0}{\beta}_1\varepsilon_7 + 3\varepsilon_8\overset{0}{\beta}_2 + \frac{1}{2}\varepsilon_8 - \overset{0}{a}_2)m+$$
$$+2\varepsilon_7\overset{0}{\beta}_1 + 2\varepsilon_8\overset{0}{\beta}_2; \quad \overset{0}{a}_{44} = \frac{1}{R^2}(2\overset{0}{\lambda}_5 m - \varepsilon_8);$$
$$\varepsilon_1 = \overset{1}{a}_1 + \overset{1}{b}_1 - \overset{0}{a}_1 - \overset{0}{b}_1, \quad \varepsilon_2 = \overset{1}{c} + \overset{1}{d} - \overset{0}{c} - \overset{0}{d}, \quad \varepsilon_3 = \overset{1}{a}_2 + \overset{1}{b}_2 - \overset{0}{a}_2 - \overset{0}{b}_2,$$
$$\varepsilon_4 = \frac{1}{R}(\overset{1}{\lambda}_1 - \frac{\overset{1}{\alpha}_2\overset{1}{\rho}_2}{\overset{1}{\rho}} - \overset{0}{\lambda}_1 + \frac{\overset{0}{\alpha}_2\overset{0}{\rho}_2}{\overset{0}{\rho}}); \quad \varepsilon_5 = \frac{1}{R}(\overset{1}{\lambda}_3 - \frac{\overset{1}{\alpha}_2\overset{1}{\rho}_1}{\overset{1}{\rho}} - \overset{0}{\lambda}_3 + \frac{\overset{0}{\alpha}_2\overset{0}{\rho}_1}{\overset{0}{\rho}});$$
$$\varepsilon_6 = \overset{1}{\mu}_1 + \overset{1}{\lambda}_5 - \overset{0}{\mu}_1 - \overset{0}{\lambda}_5, \quad \varepsilon_7 = \overset{1}{\mu}_3 + \overset{1}{\lambda}_5 - \overset{0}{\mu}_3 + \overset{0}{\lambda}_5,$$
$$\varepsilon_8 = \overset{1}{\mu}_2 + \overset{1}{\lambda}_5 - \overset{0}{\mu}_2 - \overset{0}{\lambda}_5, \quad \varepsilon_9 = \frac{1}{R}(\overset{1}{\lambda}_2 + \frac{\overset{1}{\alpha}_2\overset{1}{\rho}_2}{\overset{1}{\rho}} - \overset{0}{\lambda}_2 - \frac{\overset{0}{\alpha}_2\overset{0}{\rho}_1}{\overset{0}{\rho}});$$

$\overset{j\,i}{\zeta}_m$ and $\overset{j\,i}{\eta}_m$ are the Fourier coefficients of the given on the boundary functions.

The unknown values $\overset{j}{X}_{mk}$ are defined from the system (14) in which we put $j = 1$.

The values of the coefficients $\overset{1}{a}_{kp}$ deffer from $\overset{0}{a}_{kp}$ only by that we have changed the signs before the expressions for $\overset{0}{a}_{kp}$ and in the expression itself instead of the symbol m we have written $(-m)$. For $m = 0$, for finding the unknown constants $\overset{j}{A}{}^i$ and $\overset{j}{B}{}^i$ appearing in (12), we obtain the systems

$$\begin{aligned}&\varepsilon_1\overset{j}{A}{}^1 + \varepsilon_2\overset{j}{A}{}^2 = \overset{j\,1}{\zeta}_0\overset{j}{c}, \quad \varepsilon_6\overset{j}{B}{}^1 + \varepsilon_7\overset{j}{B}{}^2 = \overset{j\,1}{\eta}_0\overset{j}{c}, \quad \overset{0}{c} = 1, \quad \overset{1}{c} = -R^2,\\ &\varepsilon_2\overset{j}{A}{}^1 + \varepsilon_3\overset{j}{A}{}^2 = \overset{j\,2}{\zeta}_0\overset{j}{c}, \quad \varepsilon_7\overset{j}{B}{}^1 + \varepsilon_8\overset{j}{B}{}^2 = \overset{j\,2}{\eta}_0\overset{j}{c}, \quad j = 0, 1.\end{aligned} \tag{15}$$

Relying on the theorems on the uniqueness of a solution of the problem we can conclude that the principal determinants of the systems (14) and (15) are different from zero.

Substituting the solutions of the systems (14) and (15) into (13) and (12), we obtain the representations $\overset{j}{w}{}^{i}_1(x)$ in the form of infinite series.

The solution $\overset{j}{w}{}^{i}_2(x)$ of the problem [(1), (11)] is constructed in the form of series [5]:

$$\overset{j}{w}{}^{i}_2(x)=\sum_{m=0}^{\infty}\Big[\alpha^i_m x+\beta^i_m r^2\,\mathrm{grad}\Big]\overset{j}{f}_{3m}(x), \tag{16}$$

where $\overset{j}{f}_{3m}(x)$ is the harmonic function which is defined by formulas (6); α^i_m and β^i_m are the unknown constants, $j=0,1$; $i=1,2$.

To find the above values, we substitute formulas (16) and (5) into (1) instead of the values $\overset{j}{u}{}^{i}(x)$ and $\overset{j}{u}_3(x)$.

Passing to the limit, as $r\to R$, for every value of the index m, we obtain the system

$$\Big[2(\overset{j}{a}_1+\overset{j}{b}_1)+\overset{j}{b}_1 m\Big]\alpha^1_m+2(\overset{j}{b}_1+2\overset{j}{a}_1)m\overset{j}{\beta}{}^1_m+\Big[2(\overset{j}{c}+\overset{j}{d})+\overset{j}{d}m\Big]\alpha^2_m+$$
$$+2(\overset{j}{d}+2\overset{j}{c})m\beta^2_m=\overset{j}{\gamma}_1,$$
$$\Big[2(\overset{j}{c}+\overset{j}{d})+\overset{j}{d}m\Big]\alpha^1_m+2(\overset{j}{d}+2\overset{j}{c})m\overset{j}{\beta}{}^1_m+\Big[2(\overset{j}{a}_2+\overset{j}{b}_2)+\overset{j}{b}_2 m\Big]\alpha^2_m+$$
$$+2(\overset{j}{b}_2+2\overset{j}{a}_2)m\beta^2_m=\overset{j}{\gamma}_2,\quad j=0,1;\quad m=1,2,...$$

Moreover, taking into account (5) and (16), from the conditions (11), for α^i_m and β^i_m we obtain two more equations:

$$(\overset{*}{\lambda}_1+\overset{*}{\mu}_1)\Big[2\alpha^1_m+m(\alpha^1_m+2\beta^1_m)\Big]+(\overset{*}{\lambda}_2+\overset{*}{\mu}_2)\Big[2\alpha^2_m+m(\alpha^2_m+2\beta^2_m)\Big]=\overset{*}{\gamma},$$
$$\overset{*}{\mu}_1(\alpha^1_m+2m\beta^1_m)+\mu^*_2(\alpha^2_m+2\beta^2_m)=0,$$

where $\overset{*}{\lambda}_1=\overset{1}{\lambda}_1+\overset{1}{\lambda}_4-\overset{0}{\lambda}_1-\overset{0}{\lambda}_4$, $\overset{*}{\lambda}_2=\overset{1}{\lambda}_3+\overset{1}{\lambda}_2-\overset{0}{\lambda}_3-\overset{0}{\lambda}_2$, $\overset{*}{\mu}_1=\overset{1}{\mu}_1+\overset{1}{\mu}_3-\overset{0}{\mu}_1-\overset{0}{\mu}_3$, $\overset{*}{\mu}_2=\overset{1}{\mu}_3+\overset{1}{\mu}_2-\overset{0}{\mu}_3-\overset{0}{\mu}_2$, $\overset{*}{\gamma}=\overset{1}{\gamma}_1+\overset{1}{\gamma}_2-\overset{0}{\gamma}_1-\overset{0}{\gamma}_2$.

Thus we have found the regular solution $\overset{j}{U}(x)=(\overset{j}{u}(x),\overset{j}{u}_3(x))$ of the above-posed problem. $\overset{j}{u}(x)$ is given in the form of the sum (9) in which $\overset{j}{w}{}^{i}_1(x)$ are defined by formulas (12), while $\overset{j}{w}{}^{i}_2(x)$ by formulas (16); $\overset{j}{u}_3(x)$ is defined by formulas (5).

The infinite series (12), (16) and (5), as well as their first derivatives (including boundary) converge absolutely and uniformly, if the inequalities

$$\left|\overset{j\,i}{w}_1(x)\right| < \frac{1}{m^3}, \quad \left|\overset{j\,i}{w}_2(x)\right| < \frac{1}{m^3}, \quad \left|\overset{j}{u}_3\right| < \frac{1}{m^3},$$
$$i = 1,2; \quad j = 0,1, \quad m = 1,2,\ldots.$$

are valid.

To fulfill these inequalities, as one can see from the formulas for calculation of coefficients of the system (14), it suffices [6] to require that for the boundary functions the conditions $f(z) \in C^3(S)$, $F(z), f_3(z), f_4(z) \in C^2(S)$are fulfilled.

ACKNOWLEDGEMENT. The designated project has been fulfilled by financial support of Georgian National Science Foundation (Grant #GNSF/ST 06/3-033). Any idea in this publication is possessed by the author and may not represent the opinion of Georgian National Science Foundation itself.

References

1. Green A. E. and T. R. Steel T. R., Constitutive equations for interacting conting continua. *Int J. Eng. Sci.* **4** (1966), No. 4, 483–500.

2. Ruszczycki J. J., Elements of the theory of mixtures. *Kiev, Naukova Dumka* 1991, 160p.(Russian).

3. Natroshvili D. G., Jagmaidze A. Ja. and Svanadze M. Zh., Some problems of the linear theory of elastic mixtures. (Russian) With Georgian and English summaries. *Tbilisi State University Press, Tbilisi* 1986, 215 pp.

4. Giorgashvili L. G. , Toradze D. I. and Tsagareli I. I., Solution of the first boundary value problems of thermoelastic mixture for a disc. *Rep. of Enlarged Session of the Seminar of Applied Mathematics* **20** (2005), No. 2, 32–36.

5. Tsagareli I. I., Solution of the boundary value problems of statics of thermoelasticity theory of binary mixture for circle and circle external domain. Tbilisi, *Appl. Math. Inf. and Mech.* **12** (2007), No. 1, 109–115.

6. Muskhelishvili N. I., Some Basic problems of the Mathematical Theory of Elasticity. *M. Izd. AN SSSR* 1954, 648p,(Russian).