DIRICHLET AND NEUMAN PROBLEMS OF STATICS OF TWO-COMPONENT MIXTURES FOR A STRETCHED SPHEROID

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Abstract

The basic boundary value problems of statics of two-component mixtures are considered for a stretched spheroid when the limiting values of partial displacement and stress vectors are given on the boundary. A solution of the considered problems is sought in the form of an analogue of Papkovich-Neyber representations. The obtained solutions are represented as absolutely and uniformly converging series.

Key words and phrases: Mixture theory, stretched spheroid, elasticity theory.

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Introduction

One of the fundamental methods of solution of spatial problems of elasticity theory is the Fourier method based on the application of a system of curvilinear components and the subsequent separation of variables in the corresponding differential equations.

When solving problems by the Fourier method, we use various representations of equilibrium equations through harmonic, biharmonic and metaharmonic functions. When solving problems for a spheroid (stretched or compressed), a solution in the Papkovich-Neyber sense seems to be the most convenient one. A problem of elasticity theory for an isotropic ellipsoid under the action of arbitrary axially symmetric forces is solved in [7]. The solutions of the first and second basic problems for an ellipsoid of rotation are obtained in [10]. An axially symmetric problem for a hollow ellipsoid of rotation is solved in [4], while a solution of a contact problem for a stretched spheroid is given in [1].

1 Some Auxiliary Formulas and Theorems

Degenerate ellipsoidal coordinates for a stretched ellipsoid of rotation are defined by the equalities [2]

$$x_1 = c \operatorname{sh} \eta \sin \vartheta \cos \varphi, \tag{1.1}$$
$$x_2 = c \operatorname{sh} \eta \sin \vartheta \sin \varphi, \tag{1.2}$$
$$x_3 = c \operatorname{ch} \eta \cos \vartheta, \tag{1.3}$$

where c is a constant coefficient, $0 \le \eta < +\infty$, $0 \le \vartheta \le \pi$, $0 \le \varphi < 2\pi$, x_1, x_2, x_3 are the Cartesian cordinates of the point x.

The coordinate surfaces are stretched ellipsoids of rotation $\eta = \text{const}$, a two-cavity hyperboloid of rotation $\vartheta = \text{const}$, and a cavity $\varphi = \text{const}$.

The operator grad written in terms of the coordinates $(\eta, \vartheta, \varphi)$ has the form

$$\operatorname{grad} = \frac{h}{c}\Big(e_\eta \frac{\partial}{\partial \eta} + e_\vartheta \frac{\partial}{\partial \vartheta} + \frac{e_\varphi}{h \sin\vartheta \operatorname{sh}\eta}\frac{\partial}{\partial \varphi}\Big), \tag{1.4}$$

where e_η, e_ϑ, e_φ are the unit orthogonal vectors:

$$e_\eta = \big(h \operatorname{ch}\eta \sin\vartheta\cos\varphi, h \operatorname{ch}\eta \sin\vartheta \sin\varphi, h \operatorname{sh}\eta\cos\vartheta\big)^\top, \tag{1.5}$$
$$e_\vartheta = \big(h \operatorname{sh}\eta \cos\vartheta\cos\varphi, h \operatorname{sh}\eta \cos\vartheta \sin\varphi, -h \operatorname{ch}\eta\sin\vartheta\big)^\top, \tag{1.6}$$
$$e_\varphi = (-\sin\varphi, \cos\varphi, 0)^\top, \quad h = (\operatorname{ch}^2\eta - \cos^2\vartheta)^{-1/2}; \tag{1.7}$$

Here the superscript $\top$ is the transposition symbol. In the sequel, by a vector we will mean a one-column matrix.

We need to give some recurrent formulas for a Legendre function which will be used in the sequel [2]:

$$\begin{gathered}
(k-m+1)\sin\vartheta P_k^{(m-1)}(\cos\vartheta) = -\cos\vartheta P_k^{(m)}(\cos\vartheta) + P_{k-1}^{(m)}(\cos\vartheta),\\
(2k+1)\sin\vartheta P_k^{(m-1)}(\cos\vartheta) = P_{k-1}^{(m)}(\cos\vartheta) - P_{k+1}^{(m)}(\cos\vartheta),\\
\sin\vartheta P_k^{(m+1)}(\cos\vartheta) = (k-m)\cos\vartheta P_k^{(m)}(\cos\vartheta) - (k+m)P_{k-1}^{(m)}(\cos\vartheta),\\
\sin\vartheta \frac{d}{d\vartheta} P_k^{(m+1)}(\cos\vartheta) = \sin\vartheta P_k^{(m+1)}(\cos\vartheta) + m\cos\vartheta P_k^{(m)}(\cos\vartheta);
\end{gathered} \tag{1.8}$$

$$\begin{gathered}
(k-m+1)\operatorname{sh}\eta P_k^{(m-1)}(\operatorname{ch}\eta) = \operatorname{ch}\eta P_k^{(m)}(\operatorname{ch}\eta) - P_{k-1}^{(m)}(\operatorname{ch}\eta),\\
(2k+1)\operatorname{sh}\eta P_k^{(m-1)}(\operatorname{ch}\eta) = P_{k+1}^{(m)}(\operatorname{ch}\eta) - P_{k-1}^{(m)}(\operatorname{ch}\eta),\\
\operatorname{sh}\eta P_k^{(m+1)}(\operatorname{ch}\eta) = (k-m)\operatorname{ch}\eta P_k^{(m)}(\operatorname{ch}\eta) - (k+m)P_{k-1}^{(m)}(\operatorname{ch}\eta),\\
\operatorname{sh}\eta \frac{d}{d\eta} P_k^{(m)}(\operatorname{ch}\eta) = \operatorname{sh}\eta P_k^{(m+1)}(\operatorname{ch}\eta) + m\operatorname{ch}\eta P_k^{(m)}(\operatorname{ch}\eta),
\end{gathered} \tag{1.9}$$

where $P_k^{(m)}(x)$ is an adjoint Legendre function of first kind. Formulas (1.9) are also valid for the a Legendre function of second kind if the argument does not belong to the interval $[-1,1]$.

Lemma 1.1 The following equalities are fulfilled for any $k\geq 0$:

$$\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)P_{k+1}^{(m)}(\operatorname{ch}\eta)Y_{k+1}^{(m)}(\vartheta,\varphi)=\sqrt{\frac{(k-m)(2k+3)}{k-m+1}}\,g_k^{(m+1)}(\eta,\vartheta,\varphi),$$

$$\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)P_{k-1}^{(m)}(\operatorname{ch}\eta)Y_{k-1}^{(m)}(\vartheta,\varphi)=-\frac{1}{c}\sqrt{\frac{4k^2-1}{(k+m)(k+m+1)}}\times$$
$$\times P_k^{(m+1)}(\operatorname{ch}\eta)Y_k^{(m+1)}(\vartheta,\varphi)+(k-m)\sqrt{\frac{2k-1}{(k+m)(k+m+1)}}\,g_k^{(m+1)}(\eta,\vartheta,\varphi),$$

$$\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)P_{k+1}^{(m)}(\operatorname{ch}\eta)Y_{k+1}^{(m)}(\vartheta,\varphi)=$$
$$=-(k-m+2)\sqrt{(2k+3)(k+m)(k+m+1)}\,g_k^{(m-1)}(\eta,\vartheta,\varphi),$$

$$\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)P_{k-1}^{(m)}(\operatorname{ch}\eta)Y_{k-1}^{(m)}(\vartheta,\varphi)=$$
$$=-\frac{1}{c}\sqrt{(4k^2-1)(k-m)(k-m+1)}\,P_k^{(m-1)}(\operatorname{ch}\eta)Y_k^{(m-1)}(\vartheta,\varphi)-$$
$$-(k-m+2)\sqrt{(2k-1)(k-m)(k-m+1)}\,g_k^{(m-1)}(\eta,\vartheta,\varphi),$$

$$\frac{\partial}{\partial x_3}\,P_{k+1}^{(m)}(\operatorname{ch}\eta)Y_{k+1}^{(m)}(\vartheta,\varphi)=$$
$$=\sqrt{(2k+3)(k-m+1)(k+m+1)}\,g_k^{(m-1)}(\eta,\vartheta,\varphi), \tag{1.10}$$

$$\frac{\partial}{\partial x_3}\,P_{k-1}^{(m)}(\operatorname{ch}\eta)Y_{k-1}^{(m)}(\vartheta,\varphi)=-\frac{1}{c}\sqrt{\frac{(4k^2-1)(k-m)}{k+m}}\times$$
$$\times P_k^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\varphi)+(k-m+1)\sqrt{\frac{(2k-1)(k-m)}{k+m}}\,g_k^{(m)}(\eta,\vartheta,\varphi),$$

where $i=\sqrt{-1}$, $Y_k^{(m)}(\vartheta,\varphi)$ is a spherical function of the form [9]

$$Y_k^{(m)}(\vartheta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\,P_k^{(m)}(\cos\vartheta)e^{im\varphi}, \tag{1.11}$$

and

$$g_k^{(\ell)}(\eta,\vartheta,\varphi)=\frac{h^2}{c}\sqrt{\frac{1}{4\pi}\cdot\frac{(k-\ell)!}{(k+\ell)!}}\,\Big[\operatorname{ch}\eta P_k^{(\ell)}(\cos\vartheta)P_{k+1}^{(\ell)}(\operatorname{ch}\eta)-$$
$$-\cos\vartheta P_{k+1}^{(\ell)}(\cos\vartheta)P_k^{(\ell)}(\operatorname{ch}\eta)\Big]e^{i\ell\varphi},\quad \ell=m-1,m,m+1. \tag{1.12}$$

Proof. We will prove the second equality of formula (1.10). For this the operator $\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}$ is rewritten in terms of ellipsoidal coordinates as

$$\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2} = \frac{h^2}{c}\,e^{i\varphi}\Big[\sin\vartheta\,\mathrm{ch}\,\eta\,\frac{\partial}{\partial\eta} + \mathrm{sh}\,\eta\cos\vartheta\,\frac{\partial}{\partial\vartheta} + \frac{i(\mathrm{ch}^2\,\eta - \cos^2\vartheta)}{\mathrm{sh}\,\eta\sin\vartheta}\,\frac{\partial}{\partial\varphi}\Big].$$

Therefore we obtain

$$\Big(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\Big)P_{k-1}^{(m)}(\mathrm{ch}\,\eta)Y_{k-1}^{(m)}(\vartheta,\varphi) =$$
$$= \frac{h^2}{c}\sqrt{\frac{2k-1}{4\pi}\cdot\frac{(k-m+1)!}{(k+m-1)!}}\,e^{i(m+1)\varphi}I_{mk}(\eta,\vartheta), \tag{1.13}$$

where

$$I_{mk}(\eta,\vartheta) = \Big[\sin\vartheta\,\mathrm{ch}\,\eta\,\frac{\partial}{\partial\eta} + \mathrm{sh}\,\eta\cos\vartheta\,\frac{\partial}{\partial\vartheta} - \frac{m(\mathrm{ch}^2\,\eta - \cos^2\vartheta)}{\mathrm{sh}\,\eta\sin\vartheta}\Big]\times$$
$$\times P_{k-1}^{(m)}(\mathrm{ch}\,\eta)P_{k-1}^{(m)}(\cos\vartheta). \tag{1.14}$$

From the recurrent relations (1.8) and (1.9) we obtain

$$\mathrm{ch}\,\eta\,\frac{d}{d\eta}\,P_{k-1}^{(m)}(\mathrm{ch}\,\eta) = \mathrm{ch}\,\eta P_{k-1}^{(m+1)}(\mathrm{ch}\,\eta) + m\,\frac{\mathrm{ch}^2\,\eta}{\mathrm{sh}\,\eta}\,P_{k-1}^{(m)}(\mathrm{ch}\,\eta), \tag{1.15}$$
$$\cos\vartheta\,\frac{d}{d\vartheta}\,P_{k-1}^{(m)}(\cos\vartheta) = \cos\vartheta P_{k-1}^{(m+1)}(\cos\vartheta) + m\,\frac{\cos^2\vartheta}{\sin\vartheta}\,P_{k-1}^{(m)}(\cos\vartheta). \tag{1.16}$$

Using these equalities in (1.14), we have

$$I_{mk}(\eta,\vartheta) = \sin\vartheta\,\mathrm{ch}\,\eta P_{k-1}^{(m)}(\cos\vartheta)P_{k-1}^{(m)}(\mathrm{ch}\,\eta)+$$
$$+\,\mathrm{sh}\,\eta\cos\vartheta P_{k-1}^{(m)}(\mathrm{ch}\,\eta)P_{k-1}^{(m+1)}(\cos\vartheta). \tag{1.17}$$

From the identities (1.8) and (1.9) we obtain

$$\mathrm{sh}\,\eta P_{k-1}^{(m)}(\mathrm{ch}\,\eta) = \tfrac{1}{k+m}\left[P_k^{(m+1)}(\mathrm{ch}\,\eta) - \mathrm{ch}\,\eta P_{k-1}^{(m+1)}(\mathrm{ch}\,\eta)\right], \tag{1.18}$$
$$\sin\vartheta P_{k-1}^{(m)}(\cos\vartheta) = \tfrac{1}{k+m}\left[\cos\vartheta P_{k-1}^{(m)}(\cos\vartheta) - P_k^{(m+1)}(\cos\vartheta)\right]. \tag{1.19}$$

Using these equalities in (1.17), we obtain

$$I_{mk}(\eta,\vartheta) = \frac{1}{k+m}\Big[\ \cos\vartheta P_{k-1}^{(m+1)}(\cos\vartheta)P_k^{(m+1)}(\mathrm{ch}\,\eta) - \tag{1.20}$$
$$-\ \mathrm{ch}\,\eta P_{k-1}^{(m+1)}(\mathrm{ch}\,\eta)P_k^{(m+1)}(\cos\vartheta)\Big]. \tag{1.21}$$

If in the latter equality we use the relations

$$P_{k-1}^{(m+1)}(\mathrm{ch}\,\eta) = \frac{1}{k+m+1}\Big[(2k+1)\,\mathrm{ch}\,\eta P_k^{(m+1)}(\mathrm{ch}\,\eta) - (k-m)P_{k+1}^{(m+1)}(\mathrm{ch}\,\eta)\Big],$$

$$P_{k-1}^{(m+1)}(\cos\vartheta)=\frac{1}{k+m+1}\Big[(2k+1)\cos\vartheta P_k^{(m+1)}(\cos\vartheta)-(k-m)P_{k+1}^{(m+1)}(\cos\vartheta)\Big],$$

then we have

$$I_{mk}(\eta,\vartheta)=-\frac{2k+1}{h^2(k+m)(k+m+1)}\,P_k^{(m+1)}(\operatorname{ch}\eta)P_k^{(m+1)}(\cos\vartheta)+\\ +\frac{k-m}{(k+m)(k+m+1)}\Big[\operatorname{ch}\eta P_{k+1}^{(m+1)}(\operatorname{ch}\eta)P_k^{(m+1)}(\cos\vartheta)-\\ -\cos\vartheta P_{k+1}^{(m+1)}(\cos\vartheta)P_k^{(m+1)}(\operatorname{ch}\eta)\Big].$$

If this value $I_{mk}(\eta,\vartheta)$ is inserted into (1.13), then, after some transformations, we obtain the equality which we were to prove.

The other equalities of formula (1.10) are proved analogously.

Lemma 1.2 The following equalities are fulfilled for any $k\geq 0$:

$$x_3\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)P_k^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\varphi)=-\sqrt{\frac{k-m}{k+m+1}}\,\operatorname{ch}\eta P_{k+1}^{(m+1)}(\operatorname{ch}\eta)\times\\ \times Y_k^{(m+1)}(\vartheta,\varphi)+c\operatorname{ch}^2\eta\sqrt{\frac{(2k+1)(k-m)}{k+m+1}}\,g_k^{(m+1)}(\eta,\vartheta,\varphi),\\ x_3\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)P_k^{(m)}(\operatorname{ch}\eta)Y_{k-1}^{(m)}(\vartheta,\varphi)=\\ =(k-m+2)\sqrt{(k+m)(k-m+1)}\,\operatorname{ch}\eta P_{k+1}^{(m-1)}(\operatorname{ch}\eta)Y_k^{(m-1)}(\vartheta,\varphi)-\\ -c\operatorname{ch}^2\eta(k-m+2)\sqrt{(2k+1)(k+m)(k-m+1)}\,g_k^{(m-1)}(\eta,\vartheta,\varphi),\\ x_3\,\frac{\partial}{\partial x_3}\,P_k^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\varphi)=-(k-m+1)\operatorname{ch}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\varphi)+\\ +c\operatorname{ch}^2\eta(k-m+1)\sqrt{2k+1}\,g_k^{(m)}(\eta,\vartheta,\varphi), \tag{1.22}$$

where the function $g_k^{(\ell)}(\eta,\vartheta,\varphi)$, $\ell=m-1,m,m+1$, is defined by formula (1.12).

Proof. We will prove the third equality of formula (1.22). For this the operator $x_3\,\frac{\partial}{\partial x_3}$ is written in terms of ellipsoidal coordinates

$$x_3\,\frac{\partial}{\partial x_3}=h^2\operatorname{ch}\eta\cos\vartheta\Big(\operatorname{sh}\eta\cos\vartheta\,\frac{\partial}{\partial\eta}-\operatorname{ch}\eta\sin\vartheta\,\frac{\partial}{\partial\vartheta}\Big).$$

Thus we obtain

$$x_3\,\frac{\partial}{\partial x_3}\,P_k^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\varphi)=h^2\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\,e^{im\varphi}I_{mk}(\eta,\vartheta), \tag{1.23}$$

where

$$I_{mk}(\eta,\vartheta)=\operatorname{ch}\eta\cos\vartheta\Big(\cos\vartheta\operatorname{sh}\eta\,\frac{\partial}{\partial\eta}-\operatorname{ch}\eta\sin\vartheta\,\frac{\partial}{\partial\vartheta}\Big)\times$$
$$\times P_k^{(m)}(\operatorname{ch}\eta)P_k^{(m)}(\cos\vartheta). \tag{1.24}$$

From the recurrent relations (1.8) and (1.9) we have

$$\operatorname{sh}\eta\,\frac{d}{d\eta}\,P_k^{(m)}(\operatorname{ch}\eta)=(k-m+1)P_{k+1}^{(m)}(\operatorname{ch}\eta)-(k+1)\operatorname{ch}\eta P_k^{(m)}(\operatorname{ch}\eta),$$
$$\sin\vartheta\,\frac{d}{d\vartheta}\,P_k^{(m)}(\cos\vartheta)=$$
$$=(k-m+1)P_{k+1}^{(m)}(\cos\vartheta)-(k+1)\cos\vartheta P_k^{(m)}(\cos\vartheta).$$

Using these equalities in (1.24), we obtain

$$I_{mk}(\eta,\vartheta)=(k-m+1)\Big\{-\frac{1}{h^2}\,\operatorname{ch}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)+$$
$$+\operatorname{ch}^2\eta\Big[\operatorname{ch}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)P_k^{(m)}(\cos\vartheta)-\cos\vartheta P_{k+1}^{(m)}(\cos\vartheta)P_k^{(m)}(\operatorname{ch}\eta)\Big]\Big\}.$$

If this equality is used in (1.23), then

$$x_3\,\frac{\partial}{\partial x_3}\,P_k^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\varphi)=-(k-m+1)\operatorname{ch}\eta\times$$
$$\times P_{k+1}^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\eta)+c(k-m+1)\sqrt{2k+1}\operatorname{ch}^2\eta g_k^{(m)}(\eta,\vartheta,\varphi),$$

The other equalities of formula (1.22) are proved analogously.

We denote by $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ the following vectors [1], [3]

$$\begin{aligned} X_{mk}(\vartheta,\varphi)&=e_3Y_k^{(m)}(\vartheta,\varphi),\\ Y_{mk}(\vartheta,\varphi)&=\frac{1}{\sqrt{2}}\,e_2Y_k^{(m-1)}(\vartheta,\varphi),\\ Z_{mk}(\vartheta,\varphi)&=\frac{1}{\sqrt{2}}\,e_1Y_k^{(m+1)}(\vartheta,\varphi), \end{aligned} \tag{1.25}$$

where $Y_k^{(m)}(\vartheta,\varphi)$ has form (1.11), $e_1=(1,-i,0)^\top$, $e_2=(1,i,0)^\top$, $e_3=(0,0,1)^\top$.

On the sphere of unit radius the set

$$\big\{X_{mk}(\vartheta,\varphi),Y_{mk}(\vartheta,\varphi),Z_{mk}(\vartheta,\varphi)\big\}_{|m|\le k,\,k=\overline{0,\infty}}$$

forms a complete orthonormalized system of vector functions in the space L_2.

Theorem 1.3 If $\partial\Omega$ is a stretched ellipsoid of rotation ($\eta = \eta_0$), then the following equalities are valid:

$$\int\limits_{\partial\Omega} hX_{mk}(\vartheta,\varphi)\,ds = 2\sqrt{\pi}\,c^2\,\mathrm{sh}\,\eta_0 \begin{cases} e_3 & for\ \ k=0,\ \ m=0,\\ 0 & for\ \ k\geq 1,\end{cases}$$
$$\int\limits_{\partial\Omega} hY_{mk}(\vartheta,\varphi)\,ds = \sqrt{2\pi}\,c^2\,\mathrm{sh}\,\eta_0 \begin{cases} e_2 & for\ \ k=0,\ \ m=1,\\ 0 & for\ \ k\geq 1,\end{cases} \tag{1.26}$$
$$\int\limits_{\partial\Omega} hZ_{mk}(\vartheta,\varphi)\,ds = \sqrt{2\pi}\,c^2\,\mathrm{sh}\,\eta_0 \begin{cases} e_1 & for\ \ k=0,\ \ m=-1,\\ 0 & for\ \ k\geq 1.\end{cases}$$

Proof. Since
$$ds = \frac{c^2}{h}\,\mathrm{sh}\,\eta_0 \sin\vartheta\;d\vartheta\,d\varphi, \tag{1.27}$$
we have
$$\int\limits_{\partial\Omega} hX_{mk}(\vartheta,\varphi)\,ds =$$
$$= c^2\,\mathrm{sh}\,\eta_0 e_3\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\int\limits_0^{2\pi} e^{im\varphi}\,d\varphi\int\limits_0^{\pi} P_k^{(m)}(\cos\vartheta)\sin\vartheta\,d\vartheta.$$

If in this equality we take into account that
$$\int\limits_0^{2\pi} e^{im\varphi}\,d\varphi = \begin{cases} 2\pi & for\ \ m=0,\\ 0 & for\ \ m\neq 0,\end{cases}$$
then we obtain
$$\int\limits_{\partial\Omega} hX_{mk}(\vartheta,\varphi)\,ds = \sqrt{\pi}\,c^2\,\mathrm{sh}\,\eta_0 e_3 \begin{cases} \int\limits_0^{\pi} P_k(\cos\vartheta)\sin\vartheta\,d\vartheta & for\ \ m=0,\\ 0 & for\ \ m\neq 0.\end{cases} \tag{1.28}$$

Here we have used the identity $P_k^{(0)}(\cos\vartheta)$=$P_k(\cos\vartheta)$ where $P_k(\cos\vartheta)$ is a Legendre function.

The following equality is valid:
$$\int\limits_{-1}^{1} P_k(x)P_{k'}(x)\,dx = \begin{cases} \dfrac{2}{2k+1} & for\ \ k'=k,\\ 0 & for\ \ k'\neq k.\end{cases} \tag{1.29}$$

Since $P_0(x) = 1$, with (1.29) taken into account it follows that

$$\int_0^\pi P_k(\cos\vartheta)\sin\vartheta\,d\vartheta = \int_{-1}^1 P_0(x)P_k(x)\,dx = \begin{cases} 2 & for\ \ k=0, \\ 0 & for\ \ k\geq 1. \end{cases}$$

Using this equality in (1.28), we obtain

$$\int_{\partial\Omega} hX_{mk}(\vartheta,\varphi)\,ds = 2\sqrt{\pi}\,c^2\,\mathrm{sh}\,\eta_0 \begin{cases} e_3 & fir\ \ k=0,\ \ m=0, \\ 0 & for\ \ k\geq 1. \end{cases}$$

The other equalities of formula (1.26) are proved analogously.

Theorem 1.4 If $\partial\Omega$ is a stretched ellipsoid of rotation ($\eta = \eta_0$), then the following equalities are fulfilled:

$$\int_{\partial\Omega} h\big[z\times X_{mk}(\vartheta,\varphi)\big]\,ds = -\frac{ic^3\sqrt{6\pi}}{3}\,\mathrm{sh}^2\,\eta_0 \begin{cases} e_1 & for\ \ k=1,\ \ m=-1, \\ e_2 & for\ \ k=1,\ \ m=1, \\ 0 & inothercases, \end{cases}$$

$$\int_{\partial\Omega} h\big[z\times Y_{mk}(\vartheta,\varphi)\big]\,ds = \frac{ic^3\sqrt{6\pi}}{3}\,\mathrm{sh}\,\eta_0 \begin{cases} -\,\mathrm{ch}\,\eta_0 e_2 & for\ \ k=1,\ \ m=1, \\ \sqrt{2}\,\mathrm{sh}\,\eta_0 e_3 & for\ \ k=1,\ \ m=0, \\ 0 & inothercases, \end{cases}$$

$$\int_{\partial\Omega} h\big[z\times Z_{mk}(\vartheta,\varphi)\big]\,ds = \frac{ic^3\sqrt{6\pi}}{3}\,\mathrm{sh}\,\eta_0 \begin{cases} \mathrm{ch}\,\eta_0 e_1 & for\ \ k=1,\ \ m=-1, \\ \sqrt{2}\,\mathrm{sh}\,\eta_0 e_3 & for\ \ k=1,\ \ m=0, \\ 0 & inothercases, \end{cases} \tag{1.30}$$

where $z = (z_1, z_2, z_3)^\top \in \partial\Omega$.

Proof. We will prove the validity of the first equality of formula (1.30). Note that the vector $z = (z_1, z_2, z_3)^\top$ can be written in the form

$$z = \frac{c}{2}\,\mathrm{sh}\,\eta_0\sin\vartheta\big(e_1e^{i\varphi} + e_2e^{-i\varphi}\big) + c\,\mathrm{ch}\,\eta_0\cos\vartheta e_3.$$

Using this equality and the identity

$$\begin{gathered} e_1\times e_2 = 2ie_3, \quad e_1\times e_3 = -ie_1, \\ e_2\times e_3 = ie_2, \quad e_j\times e_j = 0, \quad j=1,2,3, \end{gathered} \tag{1.31}$$

we obtain

$$z\times X_{mk}(\vartheta,\varphi) = \frac{ic}{2}\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\times$$

$$\times \operatorname{sh}\eta_0 \sin\vartheta\big[-e_1 e^{i(m+1)\varphi} + e_2 e^{i(m-1)\varphi}\big] P_k^{(m)}(\cos\vartheta). \tag{1.32}$$

From the recurrent relations (1.8) it follows that

$$\begin{aligned}
\sin\vartheta P_k^{(m)}(\cos\vartheta) &= \frac{1}{2k+1}\big[P_{k-1}^{(m+1)}(\cos\vartheta) - P_{k+1}^{(m+1)}(\cos\vartheta)\big],\\
\sin\vartheta P_k^{(m)}(\cos\vartheta) &= \frac{1}{2k+1}\Big[-(k+m)(k+m-1)P_{k-1}^{(m-1)}(\cos\vartheta)+\\
&\quad +(k-m+1)(k-m+2)P_{k+1}^{(m-1)}(\cos\vartheta)\Big].
\end{aligned} \tag{1.33}$$

Since

$$\int_0^{2\pi} e^{i(m\pm1)\varphi}\,d\varphi = \begin{cases} 2\pi & for\ \ m = \mp1,\\ 0 & for\ \ m \neq \mp1,\end{cases} \tag{1.34}$$

$$\int_0^{\pi} P_{k-1}^{(0)}(\cos\vartheta)\sin\vartheta\,d\vartheta = \int_{-1}^{1} P_0(x)P_{k-1}(x)\,dx = \begin{cases} 2 & for\ \ k = 1,\\ 0 & for\ \ k \neq 1,\end{cases} \tag{1.35}$$

$$\int_0^{\pi} P_{k+1}^{(0)}(\cos\vartheta)\sin\vartheta\,d\vartheta = \int_{-1}^{1} P_0(x)P_{k+1}(x)\,dx = 0 \quad for\ \ k \geq 0, \tag{1.36}$$

from (1.32), taking into account equalities (1.27) and (1.33), we obtain

$$\int_{\partial\Omega} h\big[z \times X_{mk}(\vartheta,\varphi)\big]\,ds = -\frac{ic^3\sqrt{6\pi}}{3}\operatorname{sh}^2\eta_0 \begin{cases} e_1 & for\ \ k=1,\ \ m=-1,\\ e_2 & for\ \ k=1,\ \ m=1,\\ 0 & otherwise.\end{cases}$$

The last two equalities of formula (1.30) are proved analogously.

Let

$$F(\vartheta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} a_{mk} Y_k^{(m)}(\vartheta,\varphi),$$

be a Laplace series in a orthonormalized system of spherical functions $Y_k^{(m)}(\vartheta,\varphi)$ [9], where

$$a_{mk} = \int_0^{2\pi} d\varphi \int_0^{\pi} F(\vartheta,\varphi)\overline{Y}_k^{(m)}(\vartheta,\varphi)\sin\vartheta\,d\vartheta,$$

here $\overline{Y}_k^{(m)}(\vartheta,\varphi)$ is the complex-conjugate function $Y_k^{(m)}(\vartheta,\varphi)$.

The next theorem is true [5].

Theorem 1.5 In order that $F(y) \in W_2^{(\ell)}(\partial\Omega)$, it is necessary and sufficient that the coefficients a_{mk} admit representations

$$a_{mk} = k^{-\ell}\beta_k^{(m)}, \quad k \geq 1,$$

where

$$\sum_{k=1}^{\infty}\sum_{m=-k}^{k} |\beta_k^{(m)}|^2 < \infty.$$

Assume that the vector $f(z)$, $z \in \partial\Omega$, satisfies the sufficient conditions under which it can be represented as a Fourier-Laplace series

$$f(z) = \sum_{k=0}^{\infty}\sum_{m=-k-1}^{k+1} \big[\alpha_{mk}X_{mk}(\vartheta,\varphi)+\beta_{mk}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}Z_{mk}(\vartheta,\varphi)\big], \tag{1.37}$$

where

$$\begin{aligned}
\alpha_{mk} &= \int_0^{2\pi} d\varphi \int_0^{\pi} f(\eta_0,\vartheta,\varphi)\overline{X}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta,\\
\beta_{mk} &= \int_0^{2\pi} d\varphi \int_0^{\pi} f(\eta_0,\vartheta,\varphi)\overline{Y}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta,\\
\gamma_{mk} &= \int_0^{2\pi} d\varphi \int_0^{\pi} f(\eta_0,\vartheta,\varphi)\overline{Z}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta.
\end{aligned} \tag{1.38}$$

Here $\overline{X}_{mk}(\vartheta,\varphi)$, $\overline{Y}_{mk}(\vartheta,\varphi)$, $\overline{Z}_{mk}(\vartheta,\varphi)$ are the complex-conjugate vectors to the vectors $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$, respectively.

If in (1.38) we take into account that $P_k^{(m)}(x) = 0$ for $m > k$, then

$$\alpha_{mk} = 0 \quad for \;\; m = -k-1, k+1, \tag{1.39}$$

$$\beta_{mk} = 0 \quad for \;\; m = -k-1, -k, \tag{1.40}$$

$$\gamma_{mk} = 0 \quad for \;\; m = k, k+1. \tag{1.41}$$

Theorem 1.5 and formulas (1.38) give rise to

Theorem 1.6 If $f(z) \in C^{\ell}(\partial\Omega)$, then the coefficients α_{mk}, β_{mk}, γ_{mk} defined by formulas (1.38) admit the following estimates:

$$\alpha_{mk} = O(k^{-\ell}), \quad \beta_{mk} = O(k^{-\ell}), \quad \gamma_{mk} = O(k^{-\ell}), \;\; \ell \geq 1. \tag{1.42}$$

Theorem 1.6 gives the sufficient condition which must be imposed on $f(z)$ so that the Fourier coefficients could have estimates (1.42).

Since [9]

$$\left|Y_k^{(m)}(\vartheta,\varphi)\right| \le \sqrt{\frac{2k+1}{4\pi}}\,,$$

the following estimates are valid for $k \ge 0$:

$$\begin{aligned} &\left|X_{mk}(\vartheta,\varphi)\right| \le \sqrt{\frac{2k+1}{4\pi}}\,, \quad \left|Y_{mk}(\vartheta,\varphi)\right| \le \sqrt{\frac{2k+1}{4\pi}}\,, \\ &\left|Z_{mk}(\vartheta,\varphi)\right| \le \sqrt{\frac{2k+1}{4\pi}}\,. \end{aligned} \tag{1.43}$$

2 Statement of the Boundary Value Problems. Uniqueness Theorems.

A system of homogeneous differential equations of statics of the spatial linear theory of mixtures of two isotropic elastic materials has the form [6], [8]

$$\begin{aligned} a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0, \\ c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{2.1}$$

where $u' = (u_1', u_2', u_3')^\top$, $u'' = (u_1'', u_2'', u_3'')^\top$ are partial displacement vectors,

$$a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_5 + \lambda_1 - \frac{\rho_2}{\rho}\,\alpha', \quad a_2 = \mu_2 - \lambda_5,$$
$$b_2 = \mu_2 + \lambda_5 + \lambda_2 + \frac{\rho_1}{\rho}\,\alpha', \quad c = \mu_3 + \lambda_5, \quad \alpha' = \lambda_3 - \lambda_4,$$
$$d = \mu_3 + \lambda_3 - \lambda_5 - \frac{\rho_1}{\rho}\,\alpha', \quad \rho = \rho_1 + \rho_2,$$

$\rho_1,\ \rho_2$ are the partial densities of the mixture; $\lambda_1, \lambda_2, \dots, \lambda_5,\ \mu_1,\ \mu_2,\ \mu_3$ are the elasticity moduli characterizing the mechanical properties of the mixture, which satisfy the conditions [6], [8]

$$\mu_1 > 0, \quad \mu_1\mu_2 - \mu_3^2 > 0, \quad \lambda_5 < 0, \quad \lambda_1 + \frac{2}{3}\,\mu_1 - \frac{\rho_2}{\rho}\,\alpha' > 0,$$
$$\Big(\lambda_1 + \frac{2}{3}\,\mu_1 - \frac{\rho_2}{\rho}\,\alpha'\Big)\Big(\lambda_2 + \frac{2}{3}\,\mu_2 + \frac{\rho_1}{\rho}\,\alpha'\Big) > \Big(\lambda_3 + \frac{2}{3}\,\mu_3 - \frac{\rho_1}{\rho}\,\alpha'\Big)^2.$$

From these inequalities it follows that [6]

$$d_1 = (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0, \quad d_2 = a_1 a_2 - c^2 > 0.$$

Let us introduce the matrix operator $T(\partial x, n)$ [6]

$$T(\partial x, n)=\begin{bmatrix} T^{(1)}(\partial x, n) & \vdots & T^{(2)}(\partial x, n) \\ \cdots & \cdots & \cdots \\ T^{(3)}(\partial x, n) & \vdots & T^{(4)}(\partial x, n) \end{bmatrix}_{6\times 6}, \tag{2.2}$$

$$T^{(\ell)}(\partial x, n)=\big[T^{(\ell)}_{kj}(\partial x, n)\big]_{3\times 3},\quad \ell=1,2,3,4,$$

$$T^{(1)}_{kj}(\partial x, n)=(\mu_1-\lambda_5)\delta_{kj}\frac{\partial}{\partial n}+\Big(\lambda_1-\frac{\rho_2}{\rho}\alpha'\Big)n_k\frac{\partial}{\partial x_j}+(\mu_1+\lambda_5)n_j\frac{\partial}{\partial x_k},$$

$$T^{(2)}_{kj}(\partial x, n)=(\mu_3+\lambda_5)\delta_{kj}\frac{\partial}{\partial n}+\Big(\lambda_3-\frac{\rho_1}{\rho}\alpha'\Big)n_k\frac{\partial}{\partial x_j}+(\mu_3-\lambda_5)n_j\frac{\partial}{\partial x_k},$$

$$T^{(3)}_{kj}(\partial x, n)=(\mu_3+\lambda_5)\delta_{kj}\frac{\partial}{\partial n}+\Big(\lambda_4+\frac{\rho_2}{\rho}\alpha'\Big)n_k\frac{\partial}{\partial x_j}+(\mu_3-\lambda_5)n_j\frac{\partial}{\partial x_k},$$

$$T^{(4)}_{kj}(\partial x, n)=(\mu_2-\lambda_5)\delta_{kj}\frac{\partial}{\partial n}+\Big(\lambda_2+\frac{\rho_1}{\rho}\alpha'\Big)n_k\frac{\partial}{\partial x_j}+(\mu_2+\lambda_5)n_j\frac{\partial}{\partial x_k},$$

where δ_{kj} is the Kronecker symbol, i.e. $\delta_{kj}=1$ for $k=j$, $\delta_{kj}=0$ for $k\neq j$, $n(x)$ is the unit vector.

The operator $T(\partial x, n)$ defined by formula (2.2) is called the generalized stress operator. The vector notation for the expressions $T^{(\ell)}(\partial x, n)u$, $\ell=1,2,3,4$, where $u=(u_1,u_2,u_3)^\top$ is a three-component vector, has the form

$$T^{(\ell)}(\partial x, n)u=\xi_\ell\frac{\partial u}{\partial n}+\eta_\ell n\operatorname{div}u+\zeta_\ell[n\times\operatorname{rot}u], \tag{2.3}$$

where

$$\begin{gathered}\xi_1=2\mu_1,\quad \eta_1=\lambda_1-\frac{\rho_2}{\rho}\alpha',\quad \zeta_1=\mu_1+\lambda_5,\quad \xi_2=\xi_3=2\mu_3,\\ \eta_2=\lambda_3-\frac{\rho_1}{\rho}\alpha',\quad \eta_3=\lambda_4+\frac{\rho_2}{\rho}\alpha',\quad \zeta_2=\zeta_3=\mu_3-\lambda_5,\\ \xi_4=2\mu_2,\quad \eta_4=\lambda_2+\frac{\rho_1}{\rho}\alpha',\quad \zeta_4=\mu_2+\lambda_5.\end{gathered} \tag{2.4}$$

Definition 2.1 *A vector $U=(u',u'')^\top$ will be called regular in a domain $\Omega\subset\mathbb{R}^3$ if u', $u''\in C^2(\Omega)\cap C^1(\overline{\Omega})$.*

Denote by Ω^+ a finite domain bounded by the stretched spheroid of rotation $\partial\Omega$; $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega}^+$, $\overline{\Omega}^+=\Omega^+\cup\partial\Omega$.

Let us formulate the following boundary value problems:

Problem. *Find, in the domain Ω^+ (Ω^-), a regular solution $U=(u',u'')^\top$ of system (2.1) which on the boundary $\partial\Omega$ satisfies one of the following conditions*

$$\textit{Dirichlet problem}:\quad [U(z)]^\pm=f(z),\ \ z\in\partial\Omega \tag{2.5}$$

or

$$\textit{Neumann problem}: \quad \big[T(\partial z, n)U(z)\big]^{\pm} = f(z), \quad z \in \partial\Omega, \tag{2.6}$$

where $f(z) = (f^{(1)}(z), f^{(2)}(z))^{\top}$ *is a six-component vector,* $f^{(j)}(z) = (f_1^{(j)}(z), f_2^{(j)}(z), f_3^{(j)}(z))^{\top}$, $j = 1, 2$, *is a three-component vector,* $n(z)$ *is the outward normal unit vector with respect to* Ω^+ *at a point* $z \in \partial\Omega$. *The operator* $T(\partial x, n)$ *is defined in* (2.2).

For the sake of brevity, we will denote by $(I)^{\pm}$ the boundary value problem with conditions (2.5), and by $(II)^{\pm}$ the problem with conditions (2.6).

In the case of problems $(I)^{-}$ and $(II)^{-}$, the vector $U(x)$ must satisfy the following conditions at infinity:

$$u_j'(x) = O(|x|^{-1}), \quad u_j''(x) = O(|x|^{-1}),$$
$$\frac{\partial u_j'(x)}{\partial x_k} = o(|x|^{-1}), \quad \frac{\partial u_j''(x)}{\partial x_k} = o(|x|^{-1}), \quad k, j = 1, 2, 3.$$

The following theorems are valid [6]

Theorem 2.2 If $\partial\Omega \in C^{1+\alpha}$, $0 < \alpha \le 1$, then problems $(I)^{\pm}$, $(II)^{-}$ admit at most one regular solution.

Theorem 2.3 If $\partial\Omega \in C^{1+\alpha}$, $0 < \alpha \le 1$, then any two regular solutions of problem $(II)^{\pm}$ may differ only in an additive vector of rigid displacement

$$u'(x) = b' + [a' \times x], \quad u''(x) = b'' + [a' \times x],$$

where a', b', b'' are arbitrary constant three-component vectors.

3 Solution of the Problem

A solution of this problem is sought in the form

$$\begin{aligned} u'(x) &= \operatorname{grad}\Phi_1(x) + \alpha_3 \operatorname{grad}\big[x_3\Phi_2(x) + \Phi_0'(x)\big] - \\ &\quad - 2e_3\big[\alpha_1\Phi_2(x) - \beta_1\Phi_3(x)\big] + \operatorname{rot}(e_3\Phi_5(x)), \\ u''(x) &= \operatorname{grad}\Phi_4(x) + \alpha_3 \operatorname{grad}\big[x_3\Phi_3(x) + \Phi_0''(x)\big] - \\ &\quad - 2e_3\big[\alpha_2\Phi_3(x) - \beta_2\Phi_2(x)\big] + \operatorname{rot}(e_3\Phi_6(x)), \end{aligned} \tag{3.1}$$

where $\Phi_0'(x)$, $\Phi_0''(x)$, $\Phi_j(x)$, $j = 1, 2, \dots, 6$, are the harmonic functions,

$$\begin{aligned} &\alpha_1 = b_2(a_1 + b_1) - d(c + d), \quad \beta_1 = da_2 - b_2c, \\ &\alpha_2 = b_1(a_2 + b_2) - d(c + d), \quad \beta_2 = da_1 - b_1c, \quad \alpha_3 = b_1b_2 - d^2. \end{aligned} \tag{3.2}$$

Let us rewrite the vectors $\operatorname{grad}\Phi_j(x)$, $j=1,4$, and $\operatorname{rot}(e_3\Phi_j(x))$, $j=5,6$, as

$$\begin{gathered}\operatorname{grad}\Phi_j(x)=\frac{1}{2}\Big[e_1\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)+\\ +e_2\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)+2e_3\,\frac{\partial}{\partial x_3}\Big]\Phi_j(x),\quad j=1,4,\\ \operatorname{rot}(e_3\Phi_j(x))=\\ =\frac{i}{2}\Big[-e_1\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)+e_2\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)\Big]\Phi_j(x),\quad j=5,6.\end{gathered}\tag{3.3}$$

We introduce the notation

$$L(\partial x)=e_1\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)+e_3\,\frac{\partial}{\partial x_3}\,,$$
$$\Psi_1(x)=\frac{1}{2}\big[\Phi_1(x)-i\Phi_5(x)\big],\quad \Psi_4(x)=\frac{1}{2}\big[\Phi_4(x)-i\Phi_6(x)\big].$$

Using this notation and taking equalities (3.3) into account, we can represent solutions (3.1) in the form

$$\begin{aligned}u'(x)&=L(\partial x)\Psi_1(x)+\alpha_3\operatorname{grad}\big[x_3\Phi_2(x)+\Phi_0'(x)\big]-\\ &\quad-2e_3\big[\alpha_1\Phi_2(x)-\beta_1\Phi_3(x)\big]+\overline{L(\partial x)\Psi_1(x)},\\ u''(x)&=L(\partial x)\Psi_4(x)+\alpha_3\operatorname{grad}\big[x_3\Phi_3(x)+\Phi_0''(x)\big]-\\ &\quad-2e_3\big[\alpha_2\Phi_3(x)-\beta_2\Phi_2(x)\big]+\overline{L(\partial x)\Psi_4(x)},\end{aligned}\tag{3.4}$$

where $L(\partial x)\Psi_j$ and $\overline{L(\partial x)\Psi_j}$, $j=1,4$ are complex-conjugate vectors, and $\Psi_j(x)$, $j=1,4$, are harmonic functions.

Functions $\Psi_j(x)$, $j=1,4$, are sought in the form

$$\begin{aligned}\Psi_j(x)=\sum_{k=0}^{\infty}\sum_{m=-k+1}^{k-1}&\Big[C_{mk}^{(j)}P_{k-1}^{(m)}(\operatorname{ch}\eta)Y_{k-1}^{(m)}(\vartheta,\varphi)+\\ &+D_{mk}^{(j)}P_{k+1}^{(m)}(\operatorname{ch}\eta)Y_{k+1}^{(m)}(\vartheta,\varphi)\Big],\quad j=1,4,\end{aligned}\tag{3.5}$$

where $P_k^{(m)}(\operatorname{ch}\eta)$ are the Legendre functions of first kind, $Y_k^{(m)}(\vartheta,\varphi)$ has form (1.11), $C_{mk}^{(j)}$, $D_{mk}^{(j)}$, $j=1,4$, are arbitrary constants.

If we choose constants $C_{mk}^{(j)}$, $D_{mk}^{(j)}$, $j=1,4$, such that

$$C_{mk}^{(j)}=-c\sqrt{\tfrac{k+m}{(4k^2-1)(k-m)}}\,A_{mk}^{(j)},\tag{3.6}$$
$$D_{mk}^{(j)}=c\sqrt{\tfrac{k-m+1}{(2k+1)(2k+3)(k+m+1)}}\,A_{mk}^{(j)},\quad j=1,4,\tag{3.7}$$

where $A^{(j)}_{mk}$, $j=1,4$, is an arbitrary constant, and take equalities (1.10) into account, then we obtain

$$L(\partial x)\Psi_j(x)=\sum_{k=0}^{\infty}\sum_{m=-k+1}^{k-1}\big[P_k^{(m)}(\operatorname{ch}\eta)X_{mk}(\vartheta,\varphi)+ \\ +\sigma^{(1)}_{mk}P_k^{(m+1)}(\operatorname{ch}\eta)Z_{mk}(\vartheta,\varphi)\big]A^{(j)}_{mk},\ \ j=1,4, \tag{3.8}$$

where

$$\sigma^{(1)}_{mk}=\sqrt{\frac{2}{(k-m)(k+m+1)}}\,.$$

From (3.8) it follows that

$$\overline{L(\partial x)\Psi_j(x)} = \textstyle\sum_{k=0}^{\infty}\sum_{m=-k+1}^{k-1}\big[P_k^{(-m)}(\operatorname{ch}\eta)\overline{X}_{-mk}(\vartheta,\varphi)+ \tag{3.9}$$
$$+\sigma^{(1)}_{-mk}P_k^{(-m+1)}(\operatorname{ch}\eta)\overline{Z}_{-mk}(\vartheta,\varphi)\big]\overline{A}^{(j)}_{-mk},\ \ j=1,4. \tag{3.10}$$

If here we take into account that

$$P_k^{(-m)}(\operatorname{ch}\eta) = (-1)^m\,\tfrac{(k-m)!}{(k+m)!}\,P_k^{(m)}(\operatorname{ch}\eta), \tag{3.11}$$
$$P_k^{(-m+1)}(\operatorname{ch}\eta) = (-1)^{m-1}\,\tfrac{(k-m+1)!}{(k+m-1)!}\,P_k^{(m-1)}(\operatorname{ch}\eta), \tag{3.12}$$

$$\overline{X}_{-mk}(\vartheta,\varphi)=(-1)^m\,X_{mk}(\vartheta,\varphi),\quad \overline{Y}_{-mk}(\vartheta,\varphi)=(-1)^{m+1}\,Z_{mk}(\vartheta,\varphi), \\ \overline{Z}_{-mk}(\vartheta,\varphi)=(-1)^{m-1}\,Y_{mk}(\vartheta,\varphi),$$

then we obtain

$$\overline{L(\partial x)\Psi_j(x)}=\sum_{k=0}^{\infty}\sum_{m=-k+1}^{k-1}\big[P_k^{(m)}(\operatorname{ch}\eta)X_{mk}(\vartheta,\varphi)+ \\ +\sigma^{(2)}_{mk}P_k^{(m-1)}(\operatorname{ch}\eta)Y_{mk}(\vartheta,\varphi)\big]B^{(j)}_{mk},\ \ j=1,4, \tag{3.13}$$

where

$$\sigma^{(2)}_{mk}=\sqrt{2(k+m)(k-m+1)},\quad B^{(j)}_{mk}=\frac{(k-m)!}{(k+m)!}\,\overline{A}^{(j)}_{-mk},\ \ j=1,4.$$

Functions $\Phi_j(x)$, $j=2,3$, are sought in the form

$$\Phi_j(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}P_k^{(m)}(\operatorname{ch}\eta)Y_k^{(m)}(\vartheta,\varphi)A^{(j)}_{mk},\ \ j=2,3, \tag{3.14}$$

where $A^{(j)}_{mk}$ are the unknown constants.

Using the representation of the operator grad (3.3) and formulas (1.22), we obtain

$$x_3 \operatorname{grad} \Phi_j(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \operatorname{ch}\eta \Big\{ -(k-m+1) P_{k+1}^{(m)}(\operatorname{ch}\eta) X_{mk}(\vartheta,\varphi) + \\ + \frac{1}{2}(k-m+2)\sigma_{mk}^{(2)} P_{k+1}^{(m-1)}(\operatorname{ch}\eta) Y_{mk}(\vartheta,\varphi) - \frac{1}{2}(k-m)\sigma_{mk}^{(1)} \times \\ \times P_{k+1}^{(m+1)}(\operatorname{ch}\eta) Z_{mk}(\vartheta,\varphi) \Big\} A_{mk}^{(j)} + c \operatorname{ch}^2\eta \operatorname{grad} \Psi_0^{(j)}(x), \quad j=2,3, \tag{3.15}$$

where

$$\Psi_0^{(j)}(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \sqrt{\frac{(2k+1)(k-m+1)}{(2k+3)(k+m+1)}} \times \\ \times P_{k+1}^{(m)}(\operatorname{ch}\eta) Y_{k+1}^{(m)}(\vartheta,\varphi) A_{mk}^{(j)}, \quad j=2,3. \tag{3.16}$$

If the functions $\Phi_0'(x)$ and $\Phi_0''(x)$ contained in (3.1) are chosen so that

$$\Phi_0'(x) = -c \operatorname{ch}^2\eta_0 \Psi_0^{(2)}(x), \quad \Phi_0''(x) = -c \operatorname{ch}^2\eta_0 \Psi_0^{(3)}(x),$$

where $\eta = \eta_0$ is the equation for the surface $\partial\Omega$, then, taking (3.15) into account, we obtain

$$\alpha_3 \operatorname{grad} \big[x_3\Phi_2(x) + \Phi_0'(x)\big] - 2e_3\big[\alpha_1\Phi_2(x) - \beta_1\Phi_3(x)\big] = \\ = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ \Big[\Big(-\alpha_3(k-m+1) \operatorname{ch}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta) + \\ + (\alpha_3 - 2\alpha_1) P_k^{(m)}(\operatorname{ch}\eta) \Big) A_{mk}^{(2)} + 2\beta_1 P_k^{(m)}(\operatorname{ch}\eta) A_{mk}^{(3)} \Big] X_{mk}(\vartheta,\varphi) + \\ + \frac{\alpha_3}{2} \operatorname{ch}\eta \Big[(k-m+2)\sigma_{mk}^{(2)} P_{k+1}^{(m-1)}(\operatorname{ch}\eta) Y_{mk}(\vartheta,\varphi) - \\ - (k-m)\sigma_{mk}^{(1)} P_{k+1}^{(m+1)}(\operatorname{ch}\eta) Z_{mk}(\vartheta,\varphi) \Big] A_{mk}^{(2)} \Big\} + \\ + c\alpha_3(\operatorname{ch}^2\eta - \operatorname{ch}^2\eta_0) \operatorname{grad} \Psi_0^{(2)}(x),$$

$$\alpha_3 \operatorname{grad} \big[x_3\Phi_3(x) + \Phi_0''(x)\big] - 2e_3\big[\alpha_2\Phi_3(x) - \beta_2\Phi_2(x)\big] = \\ = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ \Big[\Big(-\alpha_3(k-m+1) \operatorname{ch}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta) + \\ + (\alpha_3 - 2\alpha_2) P_k^{(m)}(\operatorname{ch}\eta) \Big) A_{mk}^{(3)} + 2\beta_2 P_k^{(m)}(\operatorname{ch}\eta) A_{mk}^{(2)} \Big] X_{mk}(\vartheta,\varphi) + \\ + \frac{\alpha_3}{2} \operatorname{ch}\eta \Big[(k-m+2)\sigma_{mk}^{(2)} P_{k+1}^{(m-1)}(\operatorname{ch}\eta) Y_{mk}(\vartheta,\varphi) -$$

$$- (k-m)\sigma^{(1)}_{mk}P^{(m+1)}_{k+1}(\operatorname{ch}\eta)Z_{mk}(\vartheta,\varphi)\Big]A^{(3)}_{mk}\Big\}+$$
$$+ c\alpha_3(\operatorname{ch}^2\eta-\operatorname{ch}^2\eta_0)\operatorname{grad}\Psi^{(3)}_0(x). \tag{3.17}$$

Taking equalities (3.8), (3.13) and (3.17) into account in (3.4), we have

$$\begin{aligned}
u'(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[u^{(1)}_{mk}(\eta)X_{mk}(\vartheta,\varphi)+v^{(1)}_{mk}(\eta)Y_{mk}(\vartheta,\varphi)+\\
&\quad + w^{(1)}_{mk}(\eta)Z_{mk}(\vartheta,\varphi)\Big]+\alpha_3 c(\operatorname{ch}^2\eta-\operatorname{ch}^2\eta_0)\operatorname{grad}\Psi^{(2)}_0(x),\\
u''(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[u^{(2)}_{mk}(\eta)X_{mk}(\vartheta,\varphi)+v^{(2)}_{mk}(\eta)Y_{mk}(\vartheta,\varphi)+\\
&\quad + w^{(2)}_{mk}(\eta)Z_{mk}(\vartheta,\varphi)\Big]+\alpha_3 c(\operatorname{ch}^2\eta-\operatorname{ch}^2\eta_0)\operatorname{grad}\Psi^{(3)}_0(x),
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
u^{(1)}_{mk}(\eta) &= P^{(m)}_k(\operatorname{ch}\eta)(A^{(1)}_{mk}+B^{(1)}_{mk})-\Big[\alpha_3(k-m+1)\operatorname{ch}\eta P^{(m)}_{k+1}(\operatorname{ch}\eta)+\\
&\quad +(2\alpha_1-\alpha_3)P^{(m)}_k(\operatorname{ch}\eta)\Big]A^{(2)}_{mk}+2\beta_1P^{(m)}_k(\operatorname{ch}\eta)A^{(3)}_{mk},\\
v^{(1)}_{mk}(\eta) &= \sigma^{(2)}_{mk}P^{(m-1)}_k(\operatorname{ch}\eta)B^{(1)}_{mk}+\frac{\alpha_3}{2}(k-m+2)\sigma^{(2)}_{mk}\operatorname{ch}\eta P^{(m-1)}_{k+1}(\operatorname{ch}\eta)A^{(2)}_{mk},\\
w^{(1)}_{mk}(\eta) &= \sigma^{(1)}_{mk}P^{(m+1)}_k(\operatorname{ch}\eta)A^{(1)}_{mk}-\frac{\alpha_3}{2}(k-m)\sigma^{(1)}_{mk}\operatorname{ch}\eta P^{(m+1)}_{k+1}(\operatorname{ch}\eta)A^{(2)}_{mk},\\
u^{(2)}_{mk}(\eta) &= P^{(m)}_k(\operatorname{ch}\eta)(A^{(4)}_{mk}+B^{(4)}_{mk})+2\beta_2P^{(m)}_k(\operatorname{ch}\eta)A^{(2)}_{mk}-\\
&\quad -\Big[(2\alpha_2-\alpha_3)P^{(m)}_k(\operatorname{ch}\eta)+\alpha_3(k-m+1)\operatorname{ch}\eta P^{(m)}_{k+1}(\operatorname{ch}\eta)\Big]A^{(3)}_{mk},\\
v^{(2)}_{mk}(\eta) &= \sigma^{(2)}_{mk}P^{(m-1)}_k(\operatorname{ch}\eta)B^{(4)}_{mk}+\frac{\alpha_3}{2}(k-m+2)\sigma^{(2)}_{mk}\operatorname{ch}\eta P^{(m-1)}_{k+1}(\operatorname{ch}\eta)A^{(3)}_{mk},\\
w^{(2)}_{mk}(\eta) &= \sigma^{(1)}_{mk}P^{(m+1)}_k(\operatorname{ch}\eta)A^{(4)}_{mk}-\frac{\alpha_3}{2}(k-m)\sigma^{(1)}_{mk}\operatorname{ch}\eta P^{(m+1)}_{k+1}(\operatorname{ch}\eta)A^{(3)}_{mk},\quad k\geq 0.
\end{aligned} \tag{3.19}$$

Let us calculate the stress vector $T(\partial x,n)U(x)$. To this end we represent it as

$$T(\partial x,n)U=\begin{bmatrix}H^{(1)}(\partial x,n)U\\ H^{(2)}(\partial x,n)U\end{bmatrix},$$

where

$$\begin{aligned}
H^{(1)}(\partial x,n)U &= T^{(1)}(\partial x,n)u'+T^{(2)}(\partial x,n)u'',\\
H^{(2)}(\partial x,n)U &= T^{(3)}(\partial x,n)u'+T^{(4)}(\partial x,n)u''.
\end{aligned} \tag{3.20}$$

Note that

$$\operatorname{div}L(\partial x)\ =\Delta,\ \ \operatorname{rot}L(\partial x)=-ie_3\Delta+i\operatorname{grad}\tfrac{\partial}{\partial x_3}, \tag{3.21}$$

where Δ is a Laplace operator.

Taking these equalities into account, from (3.4) we obtain

$$\begin{aligned}
&\operatorname{div} u'(x) = 2\,\frac{\partial}{\partial x_3}\big[(\alpha_3-\alpha_1)\Phi_2(x)+\beta_1\Phi_3(x)\big],\\
&\operatorname{div} u''(x) = 2\,\frac{\partial}{\partial x_3}\big[(\alpha_3-\alpha_2)\Phi_3(x)+\beta_2\Phi_2(x)\big], \qquad (3.22)\\
&\operatorname{rot} u'(x) = i\operatorname{grad}\frac{\partial}{\partial x_3}\big[\Psi_1(x)-\overline{\Psi}_1(x)\big]-2\operatorname{rot}\big[e_3\big(\alpha_1\Phi_2(x)-\beta_1\Phi_3(x)\big)\big],\\
&\operatorname{rot} u''(x) = i\operatorname{grad}\frac{\partial}{\partial x_3}\big[\Psi_4(x)-\overline{\Psi}_4(x)\big]-2\operatorname{rot}\big[e_3\big(\alpha_2\Phi_3(x)-\beta_2\Phi_2(x)\big)\big].
\end{aligned}$$

Let us rewrite the normal unit vector e_η in the form

$$e_\eta = \frac{h}{2}\,\operatorname{ch}\eta\sin\vartheta(e_1e^{i\varphi}+e_2e^{-i\varphi})+h\operatorname{sh}\eta\cos\vartheta e_3. \qquad (3.23)$$

Taking the recurrent relations for the Legendre functions (1.8) and (1.9) into account, we obtain

$$\begin{aligned}
&\frac{h}{2}\,\operatorname{ch}\eta\sin\vartheta e^{i\varphi}\,\frac{\partial\Phi_j(x)}{\partial x_3}\,e_1 = -\frac{h}{2c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}(k-m)(k-m+1)\sigma_{mk}^{(1)}\operatorname{ch}\eta\times\\
&\times P_{k+1}^{(m)}(\operatorname{ch}\eta)Z_{mk}(\vartheta,\varphi)A_{mk}^{(j)}+\frac{1}{2}\,e_1h\operatorname{sh}\eta\operatorname{ch}\eta\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)\Psi_0^{(j)}(x),\\
&\frac{h}{2}\,\operatorname{ch}\eta\sin\vartheta e^{-i\varphi}\,\frac{\partial\Phi_j(x)}{\partial x_3}\,e_2 = \frac{h}{2c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\sigma_{mk}^{(2)}\operatorname{ch}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)\times\\
&\times Y_{mk}(\vartheta,\varphi)A_{mk}^{(j)}+\frac{1}{2}\,e_2h\operatorname{sh}\eta\operatorname{ch}\eta\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)\Psi_0^{(j)}(x),\\
&h\operatorname{sh}\eta\cos\vartheta\,\frac{\partial\Phi_j(x)}{\partial x_3}\,e_3 = -\frac{h}{c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}(k-m+1)\operatorname{sh}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)\times\\
&\times X_{mk}(\vartheta,\varphi)A_{mk}^{(j)}+h\operatorname{sh}\eta\operatorname{ch}\eta e_3\,\frac{\partial\Psi_0^{(j)}(x)}{\partial x_3}\,,\quad j=2,3.
\end{aligned}$$

Using these equalities in (3.22), we have

$$\begin{aligned}
&n(x)\operatorname{div} u^j(x) = e_\eta\operatorname{div} u^j(x) = \tfrac{h}{c}\textstyle\sum_{k=0}^{\infty}\sum_{m=-k}^{k}P_{k+1}^{(m)}(\operatorname{ch}\eta)\Big[-2(k-\\
&-m+1)\operatorname{sh}\eta X_{mk}(\vartheta,\varphi)+\sigma_{mk}^{(2)}\operatorname{ch}\eta Y_{mk}(\vartheta,\varphi)-(k-m)(k-m+\\
&+1)\sigma_{mk}^{(1)}\operatorname{ch}\eta Z_{mk}(\vartheta,\varphi)\Big]\Big((\alpha_3-\alpha_j)A_{mk}^{(j+1)}+\beta_jA_{mk}^{(4-j)}\Big)+\\
&+2h\operatorname{sh}\eta\operatorname{ch}\eta\operatorname{grad}\Big[(\alpha_3-\alpha_j)\Psi_0^{(j+1)}(x)+\beta_j\Psi_0^{(4-j)}(x)\Big],\quad j=1,2. \qquad (3.24)
\end{aligned}$$

Here we have introduced the notation

$$u^{(j)}(x)=\begin{cases}u'(x) & for \ \ j=1,\\ u''(x) & for \ \ j=2.\end{cases}$$

If we use the representation of grad from (3.3) and formula (3.23), then we obtain

$$e_\eta\times i\,\mathrm{grad} = \frac{h}{2c}\Big\{2ie_3\eta\,\frac{\partial}{\partial\varphi}+e_2e^{-i\varphi}\Big[\frac{\partial}{\partial\vartheta}-i\vartheta\,\frac{\partial}{\partial\varphi}\Big]- -e_1e^{i\varphi}\Big[\frac{\partial}{\vartheta}+i\vartheta\frac{\partial}{\partial\varphi}\Big]\Big\}. \quad (3.25)$$

Equality (1.10) implies

$$\frac{\partial}{\partial x_3}\,\Psi_j(x)=\sum_{k=0}^{\infty}\sum_{m=-k+1}^{k-1}P_k^{(m)}(\mathrm{ch}\,\eta)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\ \ j=1,4. \quad (3.26)$$

Using the recurrent relations for the Legendre polynomials (1.8), we obtain

$$e^{-i\varphi}\Big[\frac{\partial}{\partial\vartheta}-i\vartheta\,\frac{\partial}{\partial\varphi}\Big]Y_k^{(m)}(\vartheta,\varphi)=-\frac{1}{\sqrt{2}}\,\sigma_{mk}^{(2)}Y_k^{(m-1)}(\vartheta,\varphi), \quad (3.27)$$

$$e^{i\varphi}\Big[\frac{\partial}{\partial\vartheta}+i\vartheta\,\frac{\partial}{\partial\varphi}\Big]Y_k^{(m)}(\vartheta,\varphi)=\frac{1}{\sqrt{2}}\,(k-m)(k+m+1)\sigma_{mk}^{(1)}Y_k^{(m+1)}(\vartheta,\varphi), \quad (3.28)$$

$$\frac{\partial}{\partial\varphi}\,Y_k^{(m)}(\vartheta,\varphi)=imY_k^{(m)}(\vartheta,\varphi). \quad (3.29)$$

Taking these equalities into account in (3.26) and (3.25), we have

$$e_\eta\times i\,\mathrm{grad}\,\frac{\partial\Psi_j(x)}{\partial x_3}=\frac{h}{2}\sum_{k=0}^{\infty}\sum_{m=-k+1}^{k-1}P_k^{(m)}(\mathrm{ch}\,\eta)\Big[-2m\eta X_{mk}(\vartheta,\varphi)- -\sigma_{mk}^{(2)}Y_{mk}(\vartheta,\varphi)-(k-m)(k+m+1)\sigma_{mk}^{(1)}Z_{mk}(\vartheta,\varphi)\Big]A_{mk}^{(j)},\ \ j=1,4. \quad (3.30)$$

Hence it follows that

$$e_\eta\times i\,\mathrm{grad}\,\frac{\partial\overline{\Psi}_j(x)}{\partial x_3}=\frac{h}{2c}\sum_{k=0}^{\infty}\sum_{m=-k+1}^{k-1}P_k^{(m)}(\mathrm{ch}\,\eta)\Big[2m\,\mathrm{ch}\,\eta X_{mk}(\vartheta,\varphi)+ +\sigma_{mk}^{(2)}Y_{mk}(\vartheta,\varphi)+(k-m)(k+m+1)\sigma_{mk}^{(1)}Z_{mk}(\vartheta,\varphi)\Big]B_{mk}^{(j)},\ \ j=1,4. \quad (3.31)$$

With formulas (1.31), (3.3) and (3.23) in hand account, we obtain

$$e_\eta\times\mathrm{rot}(e_3\Phi_j(x))=$$

$$= -\frac{h}{2c}\left\{ \operatorname{ch}\eta\sin\vartheta e_3\Big[e^{-i\varphi}\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)+e^{i\varphi}\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)\Big]- \right.$$

$$\left. -\operatorname{sh}\eta\cos\vartheta\Big[e_1\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)+e_2\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)\Big]\right\}\Phi_j(x), j=2,3. \quad (3.32)$$

By formulas (1.22) and (1.10) we have

$$\tfrac{h}{2}\operatorname{sh}\eta\cos\vartheta e_1\Big(\tfrac{\partial}{\partial x_1}+i\,\tfrac{\partial}{\partial x_2}\Big)\Phi_j(x)=\tfrac{h}{2c}\,\eta x_3 e_1\Big(\tfrac{\partial}{\partial x_1}+i\,\tfrac{\partial}{\partial x_2}\Big)\Phi_j(x)=$$

$$=-\tfrac{h}{2c}\textstyle\sum_{k=0}^{\infty}\sum_{m=-k}^{k}(k-m)\sigma_{mk}^{(1)}\operatorname{sh}\eta P_{k+1}^{(m+1)}(\operatorname{ch}\eta)Z_{mk}(\vartheta,\varphi)A_{mk}^{(j)}+$$

$$+\tfrac{h}{2}\operatorname{sh}\eta\operatorname{ch}\eta e_1\Big(\tfrac{\partial}{\partial x_1}+i\,\tfrac{\partial}{\partial x_2}\Big)\Psi_0^{(j)}(x), \quad (3.33)$$

$$\tfrac{h}{2}\operatorname{sh}\eta\cos\vartheta e_2\Big(\tfrac{\partial}{\partial x_1}-i\,\tfrac{\partial}{\partial x_2}\Big)\Phi_j(x)=\tfrac{h}{2c}\,\eta x_3 e_2\Big(\tfrac{\partial}{\partial x_1}-i\,\tfrac{\partial}{\partial x_2}\Big)\Phi_j(x)=$$

$$=-\tfrac{h}{2c}\textstyle\sum_{k=0}^{\infty}\sum_{m=-k}^{k}(k-m+2)\sigma_{mk}^{(2)}\operatorname{sh}\eta P_{k+1}^{(m-1)}(\operatorname{ch}\eta)\times$$

$$\times Y_{mk}(\vartheta,\varphi)A_{mk}^{(j)}+\tfrac{h}{2}\operatorname{sh}\eta\operatorname{ch}\eta e_2\Big(\tfrac{\partial}{\partial x_1}-i\,\tfrac{\partial}{\partial x_2}\Big)\Psi_0^{(j)}(x),\quad j=2,3;$$

$$\tfrac{h}{2}\operatorname{ch}\eta\sin\vartheta e_3\Big[e^{-i\varphi}\Big(\tfrac{\partial}{\partial x_1}+i\,\tfrac{\partial}{\partial x_2}\Big)+e^{i\varphi}\Big(\tfrac{\partial}{\partial x_1}-i\,\tfrac{\partial}{\partial x_2}\Big)\Big]\Phi_j(x)=$$

$$=\tfrac{h}{c}\,e_3\Big(\tfrac{\partial}{\partial\eta}-\eta x_3\,\tfrac{\partial}{\partial x_3}\Big)\Phi_j(x),\quad j=2,3. \quad (3.34)$$

We have used here the formulas

$$\frac{\partial}{\partial x_1}\pm i\,\frac{\partial}{\partial x_2}=$$

$$=\frac{h^2}{c}\,e^{\pm i\varphi}\Big[\sin\vartheta\operatorname{ch}\eta\,\frac{\partial}{\partial\eta}+\operatorname{sh}\eta\cos\vartheta\,\frac{\partial}{\partial\vartheta}\pm i\,\frac{\operatorname{ch}^2\eta-\cos^2\vartheta}{\sin\vartheta\operatorname{sh}\eta}\,\frac{\partial}{\partial\varphi}\Big],$$

$$\frac{\partial}{\partial x_3}=\frac{h^2}{c}\,\Big(\cos\vartheta\operatorname{sh}\eta\,\frac{\partial}{\partial\eta}-\operatorname{ch}\eta\sin\vartheta\,\frac{\partial}{\partial\vartheta}\Big).$$

Substituting the value of the function $\Phi_j(x)$, $j=2,3$, from (3.14) into (3.34) and taking into account the last equality of formula (1.22), we obtain

$$\frac{h}{2}\operatorname{ch}\eta\sin\vartheta e_3\Big[e^{-i\varphi}\Big(\frac{\partial}{\partial x_1}+i\,\frac{\partial}{\partial x_2}\Big)+e^{i\varphi}\Big(\frac{\partial}{\partial x_1}-i\,\frac{\partial}{\partial x_2}\Big)\Big]\Phi_j(x)=$$

$$=\frac{h}{c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[\frac{d}{\eta}\,P_k^{(m)}(\operatorname{ch}\eta)+(k-m+1)\operatorname{sh}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)\Big]X_{mk}(\vartheta,\varphi)A_{mk}^{(j)}-$$

$$-e_3 c\operatorname{sh}\eta\operatorname{ch}\eta\,\frac{\partial}{\partial x_3}\,\Psi_0^{(j)}(x),\quad j=2,3. \quad (3.35)$$

Using formulas (3.33) and (3.35) in (3.32), we get

$$e_\eta\times\operatorname{rot}(e_3\Phi_j(x))=\frac{h}{2c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{2\Big[\frac{d}{\eta}\,P_k^{(m)}(\operatorname{ch}\eta)+(k-m+1)\operatorname{sh}\eta\times$$

$$\times P_{k+1}^{(m)}(\operatorname{ch}\eta)\Big]X_{mk}(\vartheta,\varphi)+(k-m+2)\sigma_{mk}^{(2)}\operatorname{sh}\eta P_{k+1}^{(m-1)}(\operatorname{ch}\eta)Y_{mk}(\vartheta,\varphi)-$$
$$-(k-m)\sigma_{mk}^{(1)}\operatorname{sh}\eta P_{k+1}^{(m+1)}(\operatorname{ch}\eta)Z_{mk}(\vartheta,\varphi)\Big\}A_{mk}^{(j)}+$$
$$+h\operatorname{sh}\eta\operatorname{ch}\eta\operatorname{grad}\Psi_0^{(j)}(x),\quad j=2,3. \tag{3.36}$$

With (3.30), (3.31) and (3.36) taken into account, from the last two equalities in formula (3.22) we obtain

$$n(x)\times\operatorname{rot}u^j(x)=e_\eta\times\operatorname{rot}u^j(x)=\frac{h}{2c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{-2\Big[m\eta\times$$
$$\times P_k^{(m)}(\operatorname{ch}\eta)\big(A_{mk}^{(3j-2)}+B_{mk}^{(3j-2)}\big)+2\Big(\frac{d}{d\eta}P_k^{(m)}(\operatorname{ch}\eta)+(k-m+1)\times$$
$$\times\operatorname{sh}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)\Big)\big(\alpha_jA_{mk}^{(j+1)}-\beta_jA_{mk}^{(4-j)}\big)\Big]X_{mk}(\vartheta,\varphi)$$
$$-\sigma_{mk}^{(2)}\Big[P_k^{(m)}(\operatorname{ch}\eta)\big(A_{mk}^{(3j-2)}+B_{mk}^{(3j-2)}\big) \tag{3.37}$$
$$+2(k-m+2)\operatorname{sh}\eta P_{k+1}^{(m-1)}(\operatorname{ch}\eta)\big(\alpha_jA_{mk}^{(j+1)}-\beta_jA_{mk}^{(4-j)}\big)\Big]\times$$
$$\times Y_{mk}(\vartheta,\varphi)-\sigma_{mk}^{(1)}\Big[(k-m)(k+m+1)P_k^{(m)}(\operatorname{ch}\eta)\big(A_{mk}^{(3j-2)}+B_{mk}^{(3j-2)}\big)-$$
$$-2(k-m)\operatorname{sh}\eta P_{k+1}^{(m+1)}(\operatorname{ch}\eta)\big(\alpha_jA_{mk}^{(j+1)}-\beta_jA_{mk}^{(4-j)}\big)\Big]Z_{mk}(\vartheta,\varphi)-$$
$$-2h\operatorname{sh}\eta\operatorname{ch}\eta\operatorname{grad}\big[\alpha_j\Psi_0^{(j+1)}(x)-\beta_j\Psi_0^{(4-j)}(x)\big]\Big\},\quad j=1,2. \tag{3.38}$$

Formulas (2.3), (3.18), (3.20), (3.24) and (3.38) allow us to define the vectors $H^{(j)}(\partial x,n)U(x)$, $j=1,2$, in the form

$$H^{(j)}(\partial x,n)U(x)=$$
$$=\frac{h}{c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[a_{mk}^{(j)}(\eta)X_{mk}(\vartheta,\varphi)+b_{mk}^{(j)}(\eta)Y_{mk}(\vartheta,\varphi)+c_{mk}^{(j)}(\eta)Z_{mk}(\vartheta,\varphi)\Big]+$$
$$+\alpha_3h(\operatorname{ch}^2\eta-\operatorname{ch}^2\eta_0)\frac{d}{d\eta}\operatorname{grad}\big[\xi_{2j-1}\Psi_0^{(2)}(x)+\xi_{2j}\Psi_0^{(3)}(x)\big],j=1,2. \tag{3.39}$$

where

$$a_{mk}^{(j)}(\eta)=\frac{d}{d\eta}\big[\xi_{2j-1}u_{mk}^{(1)}(\eta)+\xi_{2j}u_{mk}^{(2)}(\eta)\big]-2(k-m+1)\operatorname{sh}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)G_{mk}^{(j)}+$$
$$+2\Big[\frac{d}{d\eta}P_k^{(m)}(\operatorname{ch}\eta)+(k-m+1)\operatorname{sh}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)\Big]H_{mk}^{(j)}-m\eta P_k^{(m)}(\operatorname{ch}\eta)E_{mk}^{(j)},$$
$$b_{mk}^{(j)}(\eta)=\frac{d}{d\eta}\big[\xi_{2j-1}v_{mk}^{(1)}(\eta)+\xi_{2j}v_{mk}^{(2)}(\eta)\big]+\sigma_{mk}^{(2)}\operatorname{ch}\eta P_{k+1}^{(m)}(\operatorname{ch}\eta)G_{mk}^{(j)}-$$

$$- 2(k-m+2)\sigma^{(2)}_{mk}\operatorname{sh}\eta P^{(m-1)}_{k+1}(\operatorname{ch}\eta)H^{(j)}_{mk} - \sigma^{(2)}_{mk}P^{(m)}_{k}(\operatorname{ch}\eta)E^{(j)}_{mk},$$

$$c^{(j)}_{mk}(\eta) = \frac{d}{d\eta}\left[\xi_{2j-1}w^{(1)}_{mk}(\eta)+\xi_{2j}w^{(2)}_{mk}(\eta)\right] \quad (3.40)$$

$$-(k-m)(k-m+1)\sigma^{(1)}_{mk}\operatorname{ch}\eta P^{(m)}_{k+1}(\operatorname{ch}\eta)\times$$
$$\times G^{(j)}_{mk}+2(k-m)\sigma^{(1)}_{mk}\operatorname{sh}\eta P^{(m+1)}_{k+1}(\operatorname{ch}\eta)H^{(j)}_{mk} \quad (3.41)$$
$$-(k-m)(k+m+1)\sigma^{(1)}_{mk}P^{(m)}_{k}(\operatorname{ch}\eta)E^{(j)}_{mk},$$

$$G^{(j)}_{mk} = \eta_{2j-1}\left[(\alpha_3-\alpha_1)A^{(2)}_{mk}+\beta_1A^{(3)}_{mk}\right]+\eta_{2j}\left[(\alpha_3-\alpha_2)A^{(3)}_{mk}+\beta_2A^{(2)}_{mk}\right],$$
$$H^{(j)}_{mk} = \zeta_{2j-1}\left[\alpha_1A^{(2)}_{mk}-\beta_1A^{(3)}_{mk}\right]+\zeta_{2j}\left[\alpha_2A^{(3)}_{mk}-\beta_2A^{(2)}_{mk}\right],$$
$$E^{(j)}_{mk} = \zeta_{2j-1}\left[A^{(1)}_{mk}+B^{(1)}_{mk}\right]+\zeta_{2j}\left[A^{(4)}_{mk}+B^{(4)}_{mk}\right], \quad j=1,2, \quad (3.42)$$

$u^{(j)}_{mk}(\eta)$, $v^{(j)}_{mk}(\eta)$, $w^{(j)}_{mk}(\eta)$, $j = 1,2$ have form (3.19). In deriving formula (3.39), we used the equalities

$$\frac{\partial}{\partial n(x)} = \frac{h}{c}\frac{\partial}{\partial\eta}, \quad (3.43)$$
$$\alpha_3(\xi_{2j-1}+\eta_{2j-1})-\alpha_1(\zeta_{2j-1}+\eta_{2j-1})+\beta_2(\zeta_{2j}+\eta_{2j}) = 0, \quad (3.44)$$
$$\alpha_3(\xi_{2j}+\eta_{2j})-\alpha_2(\zeta_{2j}+\eta_{2j})+\beta_1(\zeta_{2j-1}+\eta_{2j-1}) = 0, \quad j=1,2. \quad (3.45)$$

Formulas (3.18) and (3.39) allow us to solve problems $(I)^+$ and $(II)^+$, respectively.

Let us expand the boundary vector functions $f^{(j)}(z)$, $j = 1,2$, into the Fourier-Laplace series

$$f^{(j)}(z) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\left[\alpha^{(j)}_{mk}X_{mk}(\vartheta,\varphi)+\beta^{(j)}_{mk}Y_{mk}(\vartheta,\varphi)+\gamma^{(j)}_{mk}Z_{mk}(\vartheta,\varphi)\right], \quad (3.46)$$

where $\alpha^{(j)}_{mk}$, $\beta^{(j)}_{mk}$, $\gamma^{(j)}_{mk}$, $j = 1,2$, are the Fourier coefficients.

If in both parts of equality (3.18) we pass to the limit as $x \to z \in \partial\Omega$ $(\eta \to \eta_0)$ and take into account the boundary condition (2.5) of problem $(I)^+$ and equality (3.46), then for the unknown constants $A^{(j)}_{mk}$, $j = 1,2,3,4$,

$B_{mk}^{(j)}$, $j=1,4$, we obtain the following system of algebraic equations

$$\begin{gathered}
P_k^{(m)}(\operatorname{ch}\eta_0)\big(A_{mk}^{(3j-2)}+B_{mk}^{(3j-2)}\big)-\Big[\alpha_3(k-m+1)\operatorname{ch}\eta_0 P_{k+1}^{(m)}(\operatorname{ch}\eta_0)+\\
+(2\alpha_j-\alpha_3)P_k^{(m)}(\operatorname{ch}\eta_0)\Big]A_{mk}^{(j+1)}+2\beta_j P_k^{(m)}(\operatorname{ch}\eta_0)A_{mk}^{(4-j)}=\alpha_{mk}^{(j)},\\
\sigma_{mk}^{(2)}P_k^{(m-1)}(\operatorname{ch}\eta_0)B_{mk}^{(3j-2)}+\\
+\frac{\alpha_3}{2}(k-m+2)\sigma_{mk}^{(2)}\operatorname{ch}\eta_0 P_{k+1}^{(m-1)}(\operatorname{ch}\eta_0)A_{mk}^{(j+1)}=\beta_{mk}^{(j)},\\
\sigma_{mk}^{(1)}P_k^{(m+1)}(\operatorname{ch}\eta_0)A_{mk}^{(3j-2)}-\\
-\frac{\alpha_3}{2}(k-m)\sigma_{mk}^{(1)}\operatorname{ch}\eta_0 P_{k+1}^{(m+1)}(\operatorname{ch}\eta_0)A_{mk}^{(j+1)}=\gamma_{mk}^{(j)},\quad j=1,2.
\end{gathered} \tag{3.47}$$

Since problem $(I)^+$ admits at most one regular solution, system (3.47) is compatible. Substituting the solution of this system into (3.18), we obtain the solution of problem $(I)^+$.

In view of the fact that for $k\to\infty$ we have the asymptotics [2]

$$\frac{P_k^{(m)}(\operatorname{ch}\eta)}{P_k^{(m)}(\operatorname{ch}\eta_0)}\sim\Big(\frac{\operatorname{ch}\eta}{\operatorname{ch}\eta_0}\Big)^k, \tag{3.48}$$

we conclude that series (3.18) and (3.39) converge absolutely and uniformly in the domain Ω^+. If $z\in\partial\Omega$, then for these series to converge (absolutely and uniformly) it is sufficient to prove the convergence of the following majorizing series

$$\sum_{k=k_0}^{\infty}\sum_{j=1}^{2}k^{5/2}\Big[|\alpha_{mk}^{(j)}|+|\beta_{mk}^{(j)}|+|\gamma_{mk}^{(j)}|\Big].$$

This series converges if we require of the Fourier coefficients to admit the following estimates as $k\to\infty$:

$$\alpha_{mk}^{(j)}=O(k^{-4}),\quad \beta_{mk}^{(j)}=O(k^{-4}),\quad \gamma_{mk}^{(j)}=O(k^{-4}),\quad j=1,2.$$

By virtue of Theorem 1.6, these estimates take place if we require of the boundary vector functions $f^{(j)}(z)$, $j=1,2$, that $f^{(j)}(z)\in C^4(\partial\Omega)$, $j=1,2$.

Thus, if $f^{(j)}(z)\in C^4(\partial\Omega)$, $j=1,2$, then the vector $U(x)=(u'(x),u''(x))^\top$ represented by formulas (3.12) is a regular solution of problem $(I)^+$.

Let us now solve problem $(II)^+$. We rewrite the boundary condition (2.6) as follows:

$$\big[H^{(1)}(\partial z,n)U(z)\big]^+=f^{(1)}(z),\quad z\in\partial\Omega, \tag{3.49}$$

$$\big[H^{(2)}(\partial z,n)U(z)\big]^+=f^{(2)}(z),\quad z\in\partial\Omega. \tag{3.50}$$

Let us expand the vector $f^{(j)}(z)$, $j=1,2$, into the Fourier series (3.33)

$$f^{(j)}(z)=\frac{h}{c}\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[\alpha_{mk}^{(j)}X_{mk}(\vartheta,\varphi)+\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi)+$$
$$+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)\Big],\ \ j=1,2, \qquad (3.51)$$

where $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=1,2$, are the Fourier coefficients.

If in both parts of equality (3.39) we pass to the limit as $x\to z\in\partial\Omega$ ($\eta\to\eta_0$), then for the unknown constants $A_{mk}^{(j)}$, $j=1,2,3,4$, $B_{mk}^{(j)}$, $j=1,4$, we obtain the following system of algebraic equations

$$a_{mk}^{(j)}(\eta_0)=\alpha_{mk}^{(j)},\quad b_{mk}^{(j)}(\eta_0)=\beta_{mk}^{(j)},\quad c_{mk}^{(j)}(\eta_0)=\gamma_{mk}^{(j)},\ \ j=1,2. \qquad (3.52)$$

where $a_{mk}^{(j)}(\eta)$, $b_{mk}^{(j)}(\eta)$, $c_{mk}^{(j)}(\eta)$, $j=1,2$ have form (3.40).

The necessary and sufficient conditions for the problem $(II)^+$ to be solvable are that the principal vector and the principal moment of external forces acting on the boundary $\partial\Omega$ be equal to zero, i.e.

$$\int_{\partial\Omega}f^{(j)}(z)\,ds=0,\ \ j=1,2,\quad \int_{\partial\Omega}\big[z\times f^{(1)}(z)+z\times f^{(2)}(z)\big]\,ds=0. \qquad (3.53)$$

Substituting the value of the vectors $f^{(j)}(z)$, $j=1,2$, from (3.51) into (3.53) and using formulas (1.26), (1.30), we obtain

$$\alpha_{00}^{(j)}=0,\quad \beta_{10}^{(j)}=0,\quad \gamma_{-10}^{(j)}=0,\ \ j=1,2,\quad \sum_{j=1}^{2}\big[\beta_{01}^{(j)}+\gamma_{01}^{(j)}\big]=0,$$
$$\sum_{j=1}^{2}\big[\operatorname{sh}\eta_0\alpha_{11}^{(j)}-\operatorname{ch}\eta_0\beta_{11}^{(j)}\big]=0,\quad \sum_{j=1}^{2}\big[\operatorname{sh}\eta_0\alpha_{-11}^{(j)}-\operatorname{ch}\eta_0\gamma_{-11}^{(j)}\big]=0. \qquad (3.54)$$

By virtue of the uniqueness theorem of problem $(II)^+$ and equality (3.54) we conclude that system (3.52) is solvable. Only nine unknown constants remain undefined. This is natural because the solution of problem $(II)^+$ is defined modulo a rigid displacement vector.

If the solution of system (3.52) is substituted into (3.18), then we obtain the solution of problem $(II)^+$.

Since we have asymptotics (3.48),the series (3.18) and (3.39) converge absolutely and uniformly in the domain Ω^+. If $z\in\partial\Omega$, then these series converge absolutely and uniformly if we prove the convergence of the majorizing series

$$\sum_{k=k_0}^{\infty}\sum_{j=1}^{2}k^{3/2}\big[|\alpha_{mk}^{(j)}|+|\beta_{mk}^{(j)}|+|\gamma_{mk}^{(j)}|\big].$$

This series converges if we require of the Fourier coefficients to admit the following estimates as $k \to \infty$:

$$\alpha_{mk}^{(j)} = O(k^{-3}), \quad \beta_{mk}^{(j)} = O(k^{-3}), \quad \gamma_{mk}^{(j)} = O(k^{-3}), \quad j = 1, 2.$$

These eastimates take place if, by virtue of Theorem 1.6, $f^{(j)}(z) \in C^3(\partial\Omega)$, $j = 1, 2$.

Thus, if $f^{(j)}(z) \in C^3(\partial\Omega)$, $j = 1, 2$, then the vector $U(x) = (u'(x), u''(x))^\top$ defined by formulas (3.18) is the solution of problem $(II)^+$.

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