# FALKNER-SKAN PROBLEMS FOR CONDUCTING FLUID WITH STRONG MAGNETIC FIELD

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*Abstract*

Falkner–Skan similarity problems for conducting fluid with the help of continuity equation and Green's function are reduced to solution of integro-differential equation.

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The solution is found in a power series of a small parameter $\varepsilon = \frac{1}{N}$, where $N$ is magnetic parameter.

Recurrent correlations are found for any approximate physical characteristics of boundary layer.

In this paper is generalized the self-similar solution of classical Falkner–Skan problem for conducting fluid and its solution is given with strong external magnetic field.

The fundamental equations of the two-dimensional steady flow of the boundary layer of an incompressible conducting fluid are:

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = U_\infty\,\frac{dU_\infty}{dx} + \nu\,\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}\,(U_\infty - u), \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

with the boundary conditions

$$\begin{aligned} y = 0, &\quad u = v = 0, \\ y \to \infty, &\quad u = U_\infty(x), \end{aligned} \tag{3}$$

where $u(x,y)$, $v(x,y)$ are velocity components in boundary layer, $U_\infty(x)$ is velocity outside the boundary layer, $B_0(x)$ is external magnetic field, $\nu$ is kinematic viscosity of fluid, $\sigma$ is conductivity coefficient of fluid, $\rho$ is density of fluid.

Determine from equation (2) cross velocity $v$ which satisfies boundary condition (3). We have

$$v = -\int_0^y \frac{\partial y}{\partial x}\, dy. \tag{4}$$

Substituting equation (4) into equation (1), we obtain

$$u\,\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\, dy = U_\infty\,\frac{dU_\infty}{dx} + \nu\,\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}\,(U_\infty - u). \tag{5}$$

Introduce new variables

$$\xi = x, \quad \eta = \frac{y}{\delta(x)}$$

and new function $f(\xi,\eta)$ via

$$u(x,y) = U_\infty(x) f(\xi,\eta), \tag{6}$$

where $\delta(x)$ is thickness of boundary layer.

Substituting (6) into (5), we have

$$\frac{\partial^2 f}{\partial \eta^2} + \left(\frac{U_\infty'\delta^2}{\nu} + \frac{1}{2}\,\frac{U_\infty(\delta^2)'}{\nu}\right)\frac{\partial f}{\partial \eta}\int_0^\eta f\, d\eta + \frac{U_\infty'\delta^2}{\nu}\,(1-f^2) +$$
$$+\frac{\sigma B_0^2\delta^2}{\rho\nu}\,(1-f) = \frac{U_\infty\delta^2}{\nu}\left(f\,\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta}\int_0^\eta \frac{\partial f}{\partial \xi}\, d\eta\right).$$

From equation (6) we see that a self-similar solution exists if $f(\xi,\eta)$ is independent from $\xi$ and coefficients

$$\frac{U_\infty'\delta^2}{\nu} = \beta, \quad \frac{U_\infty(\delta^2)'}{\nu} = \overline{\beta}\,, \quad \frac{\sigma B_0^2\delta^2}{\nu} = N \tag{7}$$

are independent from $x$, that is, $\beta$, $\overline{\beta}$, $N$ are constants; so, when there is a self-similar solution, the total differential equation for $f(\eta)$ is

$$f'' + \left(\beta + \frac{1}{2}\,\overline{\beta}\right) f' \int_0^\eta f\, d\eta + \beta(1-f^2) + N(1-f) = 0 \tag{8}$$

and boundary conditions are

$$f(0) = 0, \quad f(\infty) = 1. \tag{$8'$}$$

This equation (8) is generalized Falkner and Skan equation magnetic field. It is integro-differential equation of $f(\eta)$.

Now we shall try to find the relation between $U_\infty(x)$ and $\delta(x)$, for the self-similar solution.

Integration of equations

$$\frac{\delta^2}{\nu}\frac{dU_\infty}{dx} = \beta, \quad \frac{U_\infty}{\nu}\frac{d\delta^2}{dx} = \overline{\beta} \tag{9}$$

gives

$$U_\infty(x) = cx^m, \quad \delta(x) = \left(\frac{\nu\beta}{cm}\right)^{\frac{1}{2}} x^{\frac{1-m}{2}},$$

where

$$m = \frac{\beta}{\beta + \overline{\beta}}, \quad \beta = \frac{m}{1-m}\overline{\beta}, \quad \overline{\beta} = \frac{1-m}{m}\beta. \tag{10}$$

External magnetic field is

$$B_0(x) = \sqrt{\frac{Nc^m}{\sigma\beta}}\, x^{\frac{m-1}{2}}.$$

From (9) and (10) we get the velocity distribution $U_\infty(x)$ and magnetic field $B_0(x)$ by which a self-similar solution in the boundary layer exists.

Equation (8) is integro-differential equation of relatively $f$. For solution of this problem we consider to case strong external magnetic field.

Introduce new variable

$$z = \sqrt{N}\,\eta$$

and new function

$$\Phi(z) = 1 - f(\eta).$$

Then the integro-differential equation (8) and boundary conditions (8′) will be transformed into integro-differential equation of $\Phi(z)$:

$$\Phi'' - \Phi = \frac{1}{N}\left\{\left(\beta + \frac{1}{2}\overline{\beta}\right)\left(\Phi'\int_0^z \Phi\, dz - z\Phi'\right) + \beta(2\Phi - \Phi')\right\} \tag{11}$$

with boundary conditions

$$\Phi(0) = 1, \quad \Phi(\infty) = 0. \tag{12}$$

Solution of this problem (11), (12) is

$$\Phi(z) = \Phi_0(z) +$$

$$+\frac{1}{N}\int_0^\infty \left[\beta(\Phi^2 - 2\Phi) - \left(\beta + \frac{1}{2}\overline{\beta}\right)\Phi'\int_0^\eta (1-\Phi)\, d\xi\right] G\, d\xi, \tag{13}$$

where $G(z,\xi)$ is Green function of following problem:

$$G'' - G = 0, \quad G(0) = 0, \quad G(\infty) = 0.$$

This solution is

$$G(z,\zeta) = \begin{cases} G_1(z,\zeta) = \frac{1}{2}\,[e^{-(\zeta+z)} - e^{-(\zeta-z)}], & 0 < z < \zeta, \\ G_2(z,\zeta) = \frac{1}{2}\,[e^{-(z+\zeta)} - e^{-(z-\zeta)}], & \zeta < z < \infty. \end{cases}$$

When external magnetic field is large, then $N$ is large and $\frac{1}{N} = \varepsilon \ll 1$ is small.

Solution of integro-differential equation (13) is introduced by power series relative of $\varepsilon$:

$$\Phi(z) = \sum_{k=0}^{\infty} \varepsilon^k \Phi_k(z).$$

Substituting this series into (13) and equalling coefficients with the same power $\varepsilon$, we have following relations

$$\Phi_0(z) = A(z),$$

$$\Phi_1(z) = \int_0^{\infty} \left[\beta(2\Phi_0 - \Phi_0^2) - \left(\beta + \frac{1}{2}\,\overline{\beta}\right)\Phi_0' \int_0^{\zeta} (1 - \Phi_0)\,d\zeta_1\right] G(z,\zeta)\,d\zeta,$$

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$$\Phi_{k+1}(z) = \int_0^{\infty} \left[\beta\left(2\Phi_0 - \sum_{m=0}^{\infty} \Phi_{k-m}\Phi_m\right) - \right.$$
$$\left. - \left(\beta + \frac{1}{2}\,\overline{\beta}\right)\left(\zeta\Phi_k' - \sum_{m=0}^{\infty} \Phi_{k-m}' \int_0^{\zeta} \Phi_m\,d\zeta_1\right)\right] G(z,\zeta)\,d\zeta,$$

where $\Phi_0(z)$ is the solution of problem

$$\Phi_0'' - \Phi_0 = 0, \quad \Phi_0(0) = 1, \quad \Phi_0(\infty) = 0.$$

That is, $\Phi_0(z) = e^{-z}$.

Limiting first and second approximations

$$\Phi(z) \approx \Phi_0(z) + \frac{1}{N}\,\Phi_1(z),$$

where

$$\Phi_1(z) = \frac{1}{6}\,\overline{\beta}(e^{-2z} - e^{-z}) - \frac{1}{4}\left[\left(\beta + \frac{1}{2}\,\overline{\beta}\right)z^2 + \left(3\beta - \frac{1}{2}\,\overline{\beta}\right)z\right]e^{-z},$$
$$\Phi_1(0) = 0, \quad \Phi_1(\infty) = 0.$$

This approximate solution makes it possible to determine all physical properties of boundary layer conducting fluid with strong magnetic field. It is needed to mention, that all this expressions are true when $N \neq 0$.

This skin friction is:

$$\frac{\delta\tau_w}{\mu U_\infty} = \sqrt{N} - \frac{1}{\sqrt{N}}\left(\frac{3}{4}\beta + \frac{1}{24}\overline{\beta}\right).$$

The displacement thickness is:

$$\delta^*(x) = \frac{\delta}{\sqrt{N}}\int_0^\infty \left(\Phi_0 + \frac{1}{N}\Phi_1\right) dz.$$

Similarly the momentum thickness is given by:

$$\delta^{**} = \frac{\delta}{\sqrt{N}}\int_0^\infty \left(\Phi_0 + \frac{1}{N}\Phi_1\right)\left(1 - \Phi_0 - \frac{1}{N}\Phi_1\right) dz.$$

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