BASIC PROBLEMS OF THE MOMENT THEORY OF ELASTICITY

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Abstract

A general representation of solutions is obtained for a system of homogeneous differential equations of statics and stationary oscillations of the moment theory of elasticity. Theorems of the uniqueness of the considered boundary value problems are proved. Solutions are obtained in terms of absolutely and uniformly convergent series.

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Introduction

The general theory of asymmetric elasticity was developed by brothers E. and F. Cosserat in 1910 [3]. In the classical theory of elasticity a material particle coincides with a point, while a deformed state is described by the displacement of the point. As different from the classical theory, each particle of the Cosserat medium is a small, absolutely solid body. Deformation of such a medium is described not only by the displacement vector u, but also by the rotation vector ω, i.e. by the value which is a function of position x and time t.

E. Aero and E. Kuvshinski [1], V. Palmov [14], V. Kupradze, T. Gegelia, M. Basheleishvili, and T. Burchuladze [9], L. Giorgashvili [5], L. Giorgashvili and K. Skvitaridze [7], V. Novatski [13], D. Natroshvili [12], and other authors devoted interesting works to the linear theory of the Cosserat medium.

In [9], the basic boundary value and contact problems of moment elasticity are solved by the method of a potential and singular integral equations when a solid body is bounded by piecewise-smooth surfaces.

In the present work, we solve the basic problems of moment elasticity for a ball and the whole space with a spherical cavity.

1. Basic Equations

Equation of the elastic-dynamic state of the medium in terms of displacement and rotation components corresponding to mass force $F(x,t)$ and mass moment $G(x,t)$ has the form [1], [4], [9]

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x,t)+(\lambda+\mu-\alpha)grad\,div\,u(x,t)\\
&+2\alpha rot\omega(x,t)+\rho F(x,t)=\rho\,\frac{\partial^2 u(x,t)}{\partial t^2},\\
&(\nu+\beta)\Delta\omega(x,t)+(\varepsilon+\nu-\beta)grad\,div\omega(x,t)\\
&+2\alpha rot\,u(x,t)-4\alpha\omega(x,t)+\rho G(x,t)=\mathcal{I}\,\frac{\partial^2\omega(x,t)}{\partial t^2}
\end{aligned}\tag{1.1}$$

where $u=(u_1,u_2,u_3)^\top$ is a displacement vector, $\omega=(\omega_1,\omega_2,\omega_3)^\top$ is a rotation vector, t is time value, ρ is medium density, Δ is the Laplace operator, $\top$ is a transposition symbol, and λ, μ, α, $\mathcal{I}$, ε, ν β are constants characterizing the physical properties of an elastic body and satisfying the conditions

$$\mu>0,\ \ 3\lambda+2\mu>0,\ \ \alpha>0,\ \ \nu>0,\ \ 3\varepsilon+2\nu>0,\ \ \beta>0. \tag{1.2}$$

Assume that the displacement and rotation components, mass forces and moments are periodic time functions, i.e. they can be written in the form $u(x,t)=u(x)\exp(-it\sigma)$, $\omega(x,t)=\omega(x)\exp(-it\sigma)$, $F(x,t)=F(x)\exp(-it\sigma)$, $G(x,t)=G(x)\exp(-it\sigma)$, where $\sigma\in R^1$ is oscillation frequency. Then from (1.1) we obtain the equation of an elastic-oscillatory state of the medium corresponding to the mass force $F(x)$, mass moment $G(x)$ and oscillation frequency σ

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)grad\,div\,u(x)\\
&+2\alpha rot\omega(x)+\rho\sigma^2u(x)+\rho F(x)=0,\\
&(\nu+\beta)\Delta\omega(x)+(\varepsilon+\nu-\beta)grad\,div\omega(x)\\
&+2\alpha rot\,u(x)+(\mathcal{I}\sigma^2-4\alpha)\omega(x)+\rho G(x)=0.
\end{aligned}\tag{1.3}$$

An equation of an elastic-static state of the medium ($\sigma=0$) corresponding to the mass force $F(x)$ and mass moment $G(x)$ has the form

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)grad\,div\,u(x)+2\alpha rot\omega(x)+\rho F(x)=0,\\
&(\nu+\beta)\Delta\omega(x)+(\varepsilon+\nu-\beta)grad\,div\omega(x)+2\alpha rot\,u(x)\\
&-4\alpha\omega(x)+\rho G(x)=0.
\end{aligned}\tag{1.4}$$

We introduce the matrix differential operator

$$T(\partial x, n(x)) = \begin{bmatrix} T^{(3)}(\partial x, n)...T^{(4)}(\partial x, n) \\ ... \\ T^{(3)}(\partial x, n)...T^{(4)}(\partial x, n) \end{bmatrix} \tag{1.5}$$

where

$$\begin{aligned} T^{(\ell)}(\partial x, n) &= \left[T^{(\ell)}_{kj}(\partial x, n)\right]_{3\times 3}, \quad \ell = 1,2,3,4, \\ T^{(1)}_{kj}(\partial x, n) &= (\mu+\alpha)\delta_{kj}\,\frac{\partial}{\partial n(x)} + \lambda n_k(x)\,\frac{\partial}{\partial x_j} + \\ &\quad +(\mu-\alpha)n_j(x)\,\frac{\partial}{\partial x_k}, \\ T^{(2)}_{kj}(\partial x, n) &= -2\alpha\sum_{\ell=1}^{3}\varepsilon_{kj\ell}n_\ell(x), \quad T^{(3)}_{kj}(\partial x, n) = 0, \\ T^{(4)}_{kj}(\partial x, n) &= (\nu+\beta)\delta_{kj}\,\frac{\partial}{\partial n(x)} + \varepsilon n_k(x)\,\frac{\partial}{\partial x_j} + \\ &\quad +(\nu-\beta)n_j(x)\,\frac{\partial}{\partial x_k}. \end{aligned} \tag{1.6}$$

Here δ_{kj} is Kronecker's symbol, $\varepsilon_{kj\ell}$ is the Levy-Civita symbol, and $n(x) = (n_1(x), n_2(x), n_3(x))^\top$ is the unit vector.

We call $T(\partial x, n)$ the stress operator (of the moment theory of elasticity).

2. Representation of General Solutions of a System of Differential Equations of Statics and Stationary Oscillations

A homogeneous equation of an elastic-static state of the medium ($F(x) = 0$, $G(x) = 0$) is written as [1], [9]

$$\begin{aligned} (\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)grad\,div\,u(x) + 2\alpha rot\omega(x) = 0, \\ (\nu+\beta)\Delta\omega(x) + (\varepsilon+\nu-\beta)grad\,div\omega(x) + \\ +2\alpha rot\,u(x) - 4\alpha\omega(x) = 0. \end{aligned} \tag{2.1}$$

Let R^3 be the three-component Euclidean space, and $\Omega \subset R^3$ the domain bounded by the surface $\partial\Omega$. Denote by $\overline{\Omega} = \Omega \cup \partial\Omega$ a three-component vector $U = (u, \omega)^\top$ having the form $(u_1, u_2, u_3, \omega_1, \omega_2, \omega_3)^\top$.

Definition 1 *The vector $U = (u, \omega)^\top$ defined in the domain $\Omega \subset R^3$ is called regular if u_k, $\omega_k \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $k = 1, 2, 3$.*

The following theorem is true [9].

Theorem 2 *A regular solution of the homogeneous equation* (2.1) *has continuous partial derivatives of any order at an arbitrary point not belonging to* $\partial\Omega$.

Theorem 3 *For the vector* $U = (u,\omega)^\top$ *to be a solution of the system of differential equations* (2.1) *in the domain* $\Omega \subset R^3$, *it is necessary and sufficient that it be representable in the form*

$$\begin{aligned}
u(x) &= grad\,\Phi_1(x) - a\,grad\,r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_2(x)+\\
&+rot\,rot\left(xr^2\,\Phi_2(x)\right)+rot(x\Phi_3(x))+\\
&+2\alpha\left[rot\,rot(x\Phi_5(x))+rot(x\Phi_6(x))\right],\\
\omega(x) &= grad\,\Phi_4(x) - rot\left[x\left(2r\frac{\partial}{\partial r}+3\right)\Phi_2(x)\right]+\\
&+\frac{1}{2}\,rot\,rot(x\Phi_3(x))-\\
&-(\mu+\alpha)\left[\lambda_2^2 rot(x\Phi_5(x)) - rot\,rot(x\Phi_6(x))\right],
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\Delta\Phi_j(x) &= 0,\ \ j=1,2,3,\\
(\Delta-\lambda_1^2)\Phi_4(x) &= 0,\ \ (\Delta-\lambda_2^2)\Phi_j(x)=0,\ \ j=5,6,\\
a &= \frac{\mu}{\lambda+2\mu},\ \ \lambda_1^2=\frac{4\alpha}{\varepsilon+2\nu},\ \ \lambda_2^2=\frac{4\alpha\mu}{(\mu+\alpha)(\nu+\beta)},\\
x &= (x_1,x_2,x_3)^\top,\ \ r=|x|,\ \ r\frac{\partial}{\partial r}=x\cdot grad.
\end{aligned}$$

Proof. From the first equation of system (2.1) we obtain

$$\mu\Delta u(x)+(\lambda+\mu)grad\,div\,u(x)=\alpha rot\left(rot\,u(x)-2\omega(x)\right). \tag{2.3}$$

According to Theorem 2.2, we can apply the operation *rot* to the second equation of system (2.1). Then, taking into account the identity $rot\,grad = 0$, we obtain

$$\alpha rot\left(rot\,u(x)-2\omega(x)\right)=-\frac{\nu+\beta}{2}\,\Delta rot\omega(x).$$

Using this equality in (2.3) we have

$$\mu\Delta u(x)+(\lambda+\mu)grad\,div\,u(x)=-\frac{\nu+\beta}{2}\,\Delta rot\omega(x). \tag{2.4}$$

Applying the operation Δrot to the first equation of system (2.1) we obtain

$$\Delta rot rot u(x) = \frac{2\alpha}{\mu+\alpha}\,\Delta rot\omega(x). \qquad (2.5)$$

If we apply the operation Δrot to the second equality of system (2.1) and take into account (2.5), then we have

$$\Delta(\Delta - \lambda_2^2) rot\omega(x) = 0, \qquad (2.6)$$

where $\lambda_2^2 = 4\alpha\mu/(\mu+\alpha)(\nu+\beta)$.

The following lemma is valid [10].

Lemma 4 *For the vector $v = (v_1, v_2, v_3)^\top$ to be a solution of the system of differential equations*

$$(\Delta + \lambda^2)v(x) = 0, \quad div\, v(x) = 0,$$

in the domain $\Omega \subset R^3$, it is necessary and sufficient to represent it in the form

$$v(x) = rot rot(x\chi_1(x)) + rot(x\chi_2(x)),$$

where λ is a constant, $x = (x_1, x_2, x_3)^\top$, $(\Delta + \lambda^2)\chi_j(x) = 0$, $j = 1, 2,$.

By virtue of this lemma, we obtain from (2.6)

$$\Delta rot\omega(x) = A\left[rot rot(x\Phi_5(x)) + rot(x\Phi_6(x))\right], \qquad (2.7)$$

where A is an arbitrary constant, $\Phi_j(x)$, $j = 5, 6$, is the scalar function satisfying the equation $(\Delta - \lambda_2^2)\Phi_j(x) = 0$, $j = 5, 6$.

Using equality (2.7) in (2.4) we have

$$\begin{aligned}\mu\Delta u(x) + (\lambda+\mu) grad\, div\, u(x) \;&=\; -\frac{\nu+\beta}{2}\,A\left[rot rot(x\Phi_5(x))\right.\\ &\qquad \left.+ rot(x\Phi_6(x))\right]. \qquad (2.8)\end{aligned}$$

The following theorem is valid [6].

Theorem 5 *For the vector $u = (u_1, u_2, u_3)^\top$ to be a solution of the system of differential equations*

$$\mu\Delta u(x) + (\lambda+\mu) grad\, div\, u(x) = 0$$

in the domain $\Omega \subset R^3$, it is necessary and sufficient that it be representable in the form

$$\begin{aligned}u(x) \;&=\; grad\,\Phi_1(x) - a\, grad\, r^2\left(r\frac{\partial}{\partial r} + 1\right)\Phi_2(x) +\\ &\quad + rot rot\left(x r^2\,\Phi_2(x)\right) + rot(x\Phi_3(x)),\end{aligned}$$

where $x = (x_1, x_2, x_3)^\top$, $r = |x|$, $r\frac{\partial}{\partial r} = x\cdot grad$, $a = \mu(\lambda+2\mu)^{-1}$, $\Delta\Phi_j(x) = 0$, $j = 1, 2, 3$.

By Theorem 2.5, a general solution of equations (2.8) is written as

$$\begin{aligned} u(x) &= grad\,\Phi_1(x) - a\,grad\,r^2\left(r\frac{\partial}{\partial r}+1\right)\Phi_2(x)+ \\ &+rot\,rot\left(xr^2\Phi_2(x)\right)+rot(x\Phi_3(x))- \\ &-\frac{\nu+\beta}{2\mu\lambda_2^2}A\left[rot\,rot(x\Phi_5(x))+rot(x\Phi_6(x))\right]. \end{aligned} \tag{2.9}$$

If we define $rot\omega$ from the first equation of system (2.1) and substitute it into the second one, then we obtain

$$\omega(x)=\frac{\varepsilon+2\nu}{4\alpha}\,grad\,div\omega(x)+\frac{\mu}{2\alpha\lambda_2^2}\,\Delta rot\,u(x)+\frac{1}{2}\,rot\,u(x). \tag{2.10}$$

Applying the operation div to the both sides of this equality, we have $(\Delta-\lambda_1^2)div\omega(x)=0$, where $\lambda_1^2=4\alpha(\varepsilon+2\nu)^{-1}$.

Let us introduce the notation

$$\frac{\varepsilon+2\nu}{4\alpha}\,div\omega(x)=\Phi_4(x). \tag{2.11}$$

It is obvious that $(\Delta-\lambda_1^2)\Phi_4(x)=0$.

Using the value of the vector $u(x)$ from (2.9) and notation (2.11) in (2.10), we have

$$\begin{aligned} \omega(x) &= grad\,\Phi_4(x) - rot\left[x\left(2r\frac{\partial}{\partial r}+3\right)\Phi_2(x)\right]+ \\ &+\frac{1}{2}\,rot\,rot(x\Phi_3(x))+ \\ &+\frac{A}{\lambda_2^4}\left[\lambda_2^2 rot(x\Phi_5(x)) - rot\,rot(x\Phi_6(x))\right]. \end{aligned} \tag{2.12}$$

Since $A=-(\mu+\alpha)\lambda_2^4$, from (2.9) and (2.12) we obtain formulas (2.2). The first part of the theorem is thereby proved.

The second part of the theorem is proved by a direct verification. Substituting the values of the vectors $u(x)$ and $\omega(x)$ from (2.2) into (2.1), we make sure that the vector $U=(u,\omega)^\top$ defined by formulas (2.2) is a solution of system (2.1).

A system of homogeneous differential equations of an elastic-oscillatory state of the medium ($F(x)=0$, $G(x)=0$) has the form

$$\begin{aligned} &(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)grad\,div\,u(x)+ \\ &+2\alpha rot\omega(x)+\rho\sigma^2 u(x)=0, \\ &(\nu+\beta)\Delta\omega(x)+(\varepsilon+\nu-\beta)grad\,div\omega(x)+ \\ &+2\alpha rot\,u(x)+(\mathcal{I}\sigma^2-4\alpha)\omega(x)=0. \end{aligned} \tag{2.13}$$

Theorem 6 *For the vector $U = (u, \omega)^\top$ to be a solution of the system of differential equations* (2.13) *in the domain $\Omega \subset R^3$, it is necessary and sufficient that it be representable in the form*

$$\begin{aligned} u(x) &= grad\,\Phi_1(x) + \sum_{j=3}^{4} \alpha_j \left[rotrot(x\Phi_j(x)) + rot(x\Phi_{j+2}(x))\right], \quad (2.14) \\ \omega(x) &= grad\,\Phi_2(x) + \sum_{j=3}^{4} \beta_j \left[rotrot(x\Phi_{j+2}(x)) + k_j^2\,rot(x\Phi_j(x))\right], \end{aligned}$$

where

$$\begin{aligned} (\Delta + k_j^2)\Phi_j(x) &= 0, \ \ j = 1,2,3,4, \quad (\Delta + k_j^2)\Phi_{j+2}(x) = 0, \ \ j = 3,4, \\ k_1^2 &= \frac{\rho\sigma^2}{\lambda + 2\mu}, \quad k_2^2 = \frac{\mathcal{I}\sigma^2 - 4\alpha}{\varepsilon + 2\nu}, \\ k_3^2 + k_4^2 &= \sigma_1^2 + \sigma_2^2 + \frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}, \quad k_3^2 k_4^2 = \sigma_1^2\sigma_2^2, \\ \alpha_j &= 2\alpha k_j^2, \ \ \beta_j = (\mu+\alpha)(k_j^2 - \sigma_1^2), \\ x &= (x_1, x_2, x_3)^\top, \quad \sigma_1^2 = \frac{\rho\sigma^2}{\mu+\alpha}, \quad \sigma_2^2 = \frac{\mathcal{I}\sigma^2 - 4\alpha}{\nu+\beta}. \end{aligned}$$

Proof. Let the vector $U = (u, \omega)^\top$ be a solution of system (2.13). If we apply the operation *div* to the both sides of equality (2.19), then we have

$$(\Delta + k_1^2)\,div\,u(x) = 0, \quad (\Delta + k_2^2)\,div\,\omega(x) = 0, \qquad (2.15)$$

where

$$k_1^2 = \frac{\rho\sigma^2}{\lambda + 2\mu}, \quad k_2^2 = \frac{\mathcal{I}\sigma^2 - 4\alpha}{\varepsilon + 2\nu}.$$

Applying the operation *rot* to equalities (2.13) we obtain

$$\begin{aligned} (\mu+\alpha)(\Delta + \sigma_1^2)rot\,u(x) + 2\alpha rotrot\omega(x) &= 0, \\ 2\alpha rotrot\,u(x) + (\nu+\beta)(\Delta + \sigma_2^2)rot\omega(x) &= 0, \end{aligned} \qquad (2.16)$$

where

$$k_1^2 = \frac{\rho\sigma^2}{\mu+\alpha}, \quad k_2^2 = \frac{\mathcal{I}\sigma^2 - 4\alpha}{\varepsilon + \beta}.$$

From (2.16) we obtain

$$(\Delta + k_3^2)(\Delta + k_4^2)rot\,u(x) = 0, \quad (\Delta + k_3^2)(\Delta + k_4^2)rot\omega(x) = 0, \qquad (2.17)$$

where

$$k_3^2 + k_4^2 = \sigma_1^2 + \sigma_2^2 + \frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)}, \quad k_3^2 k_4^2 = \sigma_1^2\sigma_2^2.$$

Let us rewrite system (2.13) as

$$u(x) = grad\,\Phi_1(x) + u'(x), \quad \omega(x) = grad\,\Phi_2(x) + \omega'(x), \tag{2.18}$$

where

$$\begin{aligned} \Phi_1(x) = -\frac{1}{k_1^2}\, div\, u(x), \quad \Phi_2(x) = -\frac{1}{k_2^2}\, div\omega(x), \\ u'(x) = \frac{1}{\sigma_1^2}\, rot rot u(x) - \frac{2\alpha}{\rho\sigma^2}\, rot\omega(x), \\ \omega'(x) = \frac{1}{\sigma_2^2}\, rot rot\omega(x) - \frac{2\alpha}{\mathcal{I}\sigma^2 - 4\alpha}\, rot u(x). \end{aligned} \tag{2.19}$$

These equalities with (2.15) and (2.17) taken into account give

$$(\Delta + k_j^2)\Phi_j(x) = 0, \;\; j = 1, 2, \quad (\Delta + k_3^2)(\Delta + k_4^2)\left[u'(x), \omega'(x)\right]^\top = 0. \tag{2.20}$$

The vectors $u'(x)$ and $\omega'(x)$ can be represented as

$$u'(x) = \sum_{j=3}^{4} u^{(j)}(x), \quad \omega'(x) = \sum_{j=3}^{4} \omega^{(j)}(x), \tag{2.21}$$

where

$$u^{(j)}(x) = \frac{(\Delta + k_\ell^2)u'(x)}{k_\ell^2 - k_j^2}, \quad \omega^{(j)}(x) = \frac{(\Delta + k_\ell^2)\omega'(x)}{k_\ell^2 - k_j^2}, \;\; j \neq \ell = 3, 4.$$

Hence, taking into account (2.19) and (2.20), we obtain

$$\begin{aligned} (\Delta + k_j^2)u^{(j)}(x) &= 0, \quad (\Delta + k_j^2)\omega^{(j)}(x) = 0, \\ div\, u^{(j)}(x) &= 0, \quad div\omega^{(j)}(x) = 0, \;\; j = 3, 4. \end{aligned} \tag{2.22}$$

Since $rot\, u = rot\, u'$ and $rot\omega = rot\omega'$, from (2.19) we have

$$\begin{aligned} (\mu + \alpha)(k_j^2 - \sigma_1^2)u^{(j)}(x) - 2\alpha rot\omega^{(j)}(x) = 0, \\ 2\alpha rot\omega^{(j)}(x) - (\nu + \beta)(k_j^2 - \sigma_2^2)\omega^{(j)}(x) = 0, \;\; j = 3, 4. \end{aligned}$$

These equalities are satisfied if the vectors $u^{(j)}(x)$ and $\omega^{(j)}(x)$, $j = 3, 4$, are chosen as follows:

$$u^{(j)}(x) = \alpha_j V^{(j)}(x), \quad \omega^{(j)}(x) = \beta_j rot\, V^{(j)}(x), \;\; j = 3, 4. \tag{2.23}$$

where

$$\begin{aligned} \alpha_j = 2\alpha k_j^2, \quad \beta_j(\mu + \alpha)(k_j^2 - \sigma_1^2), \\ (\Delta + k_j^2)V^{(j)}(x) = 0, \quad div\, V^{(j)}(x) = 0, \;\; j = 3, 4. \end{aligned}$$

Substituting the values of the vectors $u^{(j)}(x)$ and $\omega^{(j)}(x)$ from (2.23) into (2.21) and (2.18) we have

$$u(x) = grad\,\Phi_1(x) + \sum_{j=3}^{4} \alpha_j V^{(j)}(x), \quad \omega(x) = grad\,\Phi_2(x) + \sum_{j=3}^{4} \beta_j rot\,V^{(j)}(x). \tag{2.24}$$

By virtue of Lemma 2.4 we obtain

$$V^{(j)}(x) = rot rot(x\Phi_j(x)) + rot(x\Phi_{j+2}(x)), \;\; j = 3, 4. \tag{2.25}$$

where

$$(\Delta + k_j^2)\Phi_j(x) = 0, \quad (\Delta + k_j^2)\Phi_{j+2}(x) = 0, \;\; j = 3, 4.$$

From (2.25) we have

$$rot\,V^{(j)}(x) = rot rot(x\Phi_{j+2}(x)) + k_j^2\,rot(x\Phi_j(x)). \tag{2.26}$$

If the values of the vectors $V^{(j)}(x)$ and $rot\,V^{(j)}(x)$ from (2.25) and (2.26) are substituted into (2.24), we obtain

$$u(x) = grad\,\Phi_1(x) + \sum_{j=3}^{4} \alpha_j \left[rot rot(x\Phi_j(x)) + rot(x\Phi_{j+2}(x))\right],$$

$$\omega(x) = grad\,\Phi_2(x) + \sum_{j=3}^{4} \beta_j \left[rot rot(x\Phi_{j+2}(x)) + k_j^2\,rot(x\Phi_j(x))\right].$$

The first part of the theorem is thereby proved. The second part is proved by a direct verification. Substituting the values of the vectors $u(x)$ and $\omega(x)$ from (2.14) into (2.13) and using the identities

$$\begin{aligned} \rho\sigma^2 + (\mu+\alpha)\alpha_j k_j^2 + 2\alpha k_j^2 \beta_j &= 0, \\ (\mathcal{I}\sigma^2 - 4\alpha)\beta_j - (\nu+\beta)\beta_j k_j^2 + 2\alpha\alpha_j &= 0, \;\; j = 3, 4, \end{aligned}$$

we make sure that the vector $U = (u, \omega)^\top$ represented by formulas (2.14) is a solution of system (2.13).

Remark 7 *If $\mathcal{I}\sigma^2 - 4\alpha > 0$, then $k_j^2 > 0$, $j = 1, 2, 3, 4$. If $\mathcal{I}\sigma^2 - 4\alpha < 0$, then $k_1^2 > 0$, $k_3^2 > 0$, $k_2^2 < 0$, $k_4^2 < 0$. In the sequel it will be assumed that $\mathcal{I}\sigma^2 - 4\alpha > 0$.*

Remark 8 *From (2.24) it follows that any regular solution of system (2.13) is representable in the form*

$$u(x) = V^{(1)}(x) + \sum_{j=3}^{4} \alpha_j V^{(j)}(x), \quad \omega(x) = V^{(2)}(x) + \sum_{j=3}^{4} \beta_j rot\,V^{(j)}(x), \tag{2.27}$$

where

$$\begin{aligned}(\Delta + k_j^2)V^{(j)}(x) &= 0, \ \ j = 1,2,3,4,\\ rot\, V^{(j)}(x) &= 0, \ \ j = 1,2, \quad div\, V^{(j)}(x) = 0, \ \ j = 3,4.\end{aligned}$$

Let $\Omega^+ \subset R^3$ be the finite domain bounded by the surface $\partial\Omega$, $\overline{\Omega}^+ = \Omega^+ \cup \partial\Omega$, $\Omega^- = R^3 \setminus \overline{\Omega}^+$.

Definition 9 *A regular in Ω^- solution $U = (u,\omega)^\top$ of system (2.13) satisfies the radiation condition if in the neighborhood $|x| = \infty$,*

$$V^{(j)}(x) = O(|x|^{-1}), \quad \frac{\partial V^{(j)}(x)}{\partial |x|} - ik_j V^{(j)}(x) = O(|x|^{-2}), \ \ j = 1,2,3,4. \tag{2.28}$$

3. Formulation of the Boundary Value Problems. Uniqueness Theorems

Problem $(M)^\pm$**.** *Find, in the domain Ω^+ (Ω^-), a regular solution $U = (u,\omega)^\top$ of system* (2.1) *satisfying, on the boundary $\partial\Omega$, one of the following conditions:*

$$[U(z)]^\pm = f(z) \ \ or \tag{3.1}$$

$$[T(\partial z, n)U(z)]^\pm = f(z) \ \ or \tag{3.2}$$

$$\begin{aligned}[n(z)\cdot u(z)]^\pm &= f_4(z),\\ [n(z)\times rot u(z)]^\pm &= f^{(1)}(z), \quad [\omega(z)]^\pm = f^{(2)}(z) \ \ or\end{aligned} \tag{3.3}$$

$$\begin{aligned}[n(z) ... \omega(z)]^\pm &= f_4(z),\\ [n(z)\times rot\omega(z)]^\pm &= f^{(1)}(z), \quad [u(z)]^\pm = f^{(2)}(z),\end{aligned} \tag{3.4}$$

where $f = (f^{(1)}, f^{(2)})$, $f^{(j)} = (f_1^{(j)}, f_2^{(j)}, f_3^{(j)})$, $j = 1,2$, $f_k^{(j)}(z)$, $f_4(z)$, $j = 1,2$, $k = 1,2,3$ are the functions given on $\partial\Omega$, $n(z)$ is the unit vector of the outward normal with respect to Ω^+ at a point $z \in \partial\Omega$.

In the case of the external problems $(M)^-$, the vector $U = (u,\omega)^\top$ must satisfy, in the neighborhood of the point $|x| = \infty$, the conditions

$$\begin{aligned}u_j(x) &= O(|x|^{-1}), \quad \omega_j(x) = o(|x|^{-1}),\\ \frac{\partial u_j(x)}{\partial x_k} &= o(|x|^{-1}), \quad \frac{\partial \omega_j(x)}{\partial x_k} = o(|x|^{-1}), \ \ k,j = 1,2,3.\end{aligned} \tag{3.5}$$

Denote by $(M \cdot I)^\pm, \dots, (M \cdot IV)^\pm$ the problems which contain the boundary conditions (3.1)–(3.4), respectively.

Problem $(\overset{\sigma}{M})^{\pm}$. *Find, in the domain* $U = (u, \omega)^\top$, *a regular solution* $\partial\Omega$ *of system* (2.13) *satisfying, on the boundary* $\partial\Omega$, *one of the conditions* (3.1)–(3.4). *In the case of the external problems, the vector* $U = (u, \omega)^\top$ *in the neighborhood of a point at infinity must satisfy the radiation condition.*

The following theorem is true [9].

Theorem 10 *If* $\partial\Omega \in \mathcal{L}_1(\alpha)$, $0 < \alpha \le 1$, *then problems* $(M \cdot I)^{\pm}$, $(M \cdot II)^{-}$ *admit at most one regular solution.*

Theorem 11 *If* $\partial\Omega \in \mathcal{L}_1(\alpha)$, $0 < \alpha \le 1$, *then any two solutions of problem* $(M \cdot II)^{+}$ *may differ from each other in an additive rigid displacement vector, i.e.*

$$u(x) = [a \times x] + b, \quad \omega(x) = a,$$

where $x = (x_1, x_2, x_3)^\top$, *a and b are arbitrary real three-component vectors.*

Theorem 12 *If* $\partial\Omega \in \mathcal{L}_1(\alpha)$, $0 < \alpha \le 1$, *then homogeneous problems* $(M \cdot III)_0^{\pm}$ *and* $(M \cdot IV)_0^{\pm}$ $(f_4 = 0,\ f^{(j)} = 0,\ j = 1, 2)$ *have only a trivial solution in the class of regular vectors.*

Proof. We introduce the matrix differential operator $M(\partial x)$

$$\begin{aligned} M(\partial x) &= \begin{bmatrix} M^{(1)}(\partial x) ... M^{(2)}(\partial x) \\ ... \\ M^{(3)}(\partial x) ... M^{(4)}(\partial x) \end{bmatrix} \\ M^{(\ell)}(\partial x) &= [M_{kj}(\partial x)]_{3\times 3}, \quad \ell = 1, 2, 3, 4, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} M_{kj}^{(1)}(\partial x) &= (\mu + \alpha)\delta_{kj}\Delta + (\lambda + \mu - \alpha)\frac{\partial^2}{\partial x_k \partial x_j}, \\ M_{kj}^{(2)}(\partial x) &= M_{kj}^{(3)}(\partial x) = -2\alpha \sum_{\ell=1}^{3} \varepsilon_{kj\ell} \frac{\partial}{\partial x_\ell}, \\ M_{kj}^{(4)}(\partial x) &= (\nu + \beta)\delta_{kj}(\Delta - 4\alpha) + (\varepsilon + \nu - \beta)\frac{\partial^2}{\partial x_k \partial x_j}, \end{aligned}$$

here Δ is the Laplace operator, δ_{kj} is Kronecker's symbol, $\varepsilon_{kj\ell}$ is the Levi-Civita symbol.

Using this notation, we can rewrite systems (2.1) and (2.13) as

$$M(\partial x)U(x) = 0, \tag{3.7}$$

$$M(\partial x)U(x) + \left(\rho\sigma^2 u(x), \mathcal{I}\sigma^2\omega(x)\right)^\top = 0. \tag{3.8}$$

Let us consider the scalar product $U \cdot M(\partial x)U'$, where $U = (u, \omega)^\top$, $U' = (u', \omega')^\top$ are six-component vectors, $M(\partial x)$ has form (3.6):

$$
\begin{aligned}
U \cdot M(\partial x)U' &= (\mu+\alpha)u \cdot \Delta u' + (\lambda+\mu-\alpha)u \cdot grad\, div\, u' + \\
&+2\alpha u \cdot rot\omega' + (\nu+\beta)\omega \cdot \Delta\omega' + \\
&+(\varepsilon+\nu-\beta)\omega \cdot grad\, div\omega' + \\
&+2\alpha\omega \cdot rot u' - 4\alpha\omega \cdot \omega'. \qquad (3.9)
\end{aligned}
$$

We obtain the identities

$$
\begin{aligned}
u \cdot \Delta v &= div(u\, div\, v) - div\, u\, div\, v + div[u \times rot v] - rot u \cdot rot v, \\
u \cdot grad\, div\, v &= div(u\, div\, v) - div\, u\, div\, v, \\
u \cdot rot v &= -dv[u \times v] + v \cdot rot u,
\end{aligned}
$$

where $u = (u_1, u_2, u_3)^\top$, $v = (v_1, v_2, v_3)^\top$ are arbitrary three-component vectors.

Using these equalities in (3.9) we obtain

$$
\begin{aligned}
U \cdot M(\partial x)U' &= div\, [(\lambda+2\mu)(u\, div\, u') + \\
&+(\varepsilon+2\nu)(\omega\, div\omega') + (\mu+\alpha)(u \times rot u') + \qquad (3.10) \\
&+(\nu+\beta)(\omega \times rot\omega') - 2\alpha(u \times \omega')] - E(U, U')
\end{aligned}
$$

where

$$
\begin{aligned}
E(U, U') &= (\lambda+2\mu) div\, u\, div\, u' + \\
&+(\varepsilon+2\nu) div\omega\, div\omega' + \mu rot u \cdot rot u' + \\
&+\alpha \left(rot u \cdot rot u' - 2\omega' \cdot rot u - 2\omega \cdot rot u' + \omega \cdot \omega' \right) + \\
&+(\nu+\beta) rot\omega \cdot rot\omega'. \qquad (3.11)
\end{aligned}
$$

It is obvious that $E(U, U') = E(U', U)$. If $U = U'$, then (3.1) implies

$$
\begin{aligned}
E(U, U) &= (\lambda+2\mu)(div u)^2 + (\varepsilon+2\nu)(div\omega)^2 + \mu(rot u)^2 + \\
&+(\nu+\beta)(rot\omega)^2 + \alpha(rot u - 2\omega)^2. \qquad (3.12)
\end{aligned}
$$

Applying the Gauss-Ostrogradski formula, from (2.10) we obtain

$$
\begin{aligned}
\int_{\Omega^+} U(x) \cdot M(\partial x)U'(x)\, dx &= \int_{\partial\Omega} [U(z)]^+ \cdot [P(\partial z, n)U'(z)]^+\, ds \\
&- \int_{\Omega^+} E(U, U')\, dx, \qquad (3.13)
\end{aligned}
$$

where

$$\begin{aligned} U\cdot P(\partial z,n)U' &= (\lambda+2\mu)(n\cdot u)\,div\,u' + (\varepsilon+2\nu)(n\cdot\omega)\,div\omega' - \\ &\quad -(\mu+\alpha)u\cdot(n\times rot\,u') - (\nu+\beta)\omega\cdot(n\times rot\omega') + \\ &\quad +2\alpha u\cdot(n\times\omega'). \end{aligned} \tag{3.14}$$

If we consider the domain Ω^- and keep in mind that the vectors $U = (u,\omega)^\top$ and $U' = (u',\omega')^\top$ satisfy condition (3.5), then in the domain Ω^- formula (3.13) takes the form

$$\begin{aligned} &\int\limits_{\Omega^-} U(x)\cdot M(\partial x)U'(x)\,dx \\ &= -\int\limits_{\partial\Omega}[U(z)]^-\cdot\left[P(\partial z,n)U'(z)\right]^-\,ds - \int\limits_{\Omega^-} E(U,U')\,dx. \end{aligned} \tag{3.15}$$

Let $U' = U$, then formulas (3.13) and (3.15) can be rewritten as

$$\begin{aligned} \int\limits_{\Omega^\pm} U(x)\cdot M(\partial x)U(x)\,dx &= \pm\int\limits_{\partial\Omega}[U(z)]^\pm\cdot[P(\partial z,n)U(z)]^\pm\,ds \\ &\quad -\int\limits_{\Omega^\pm} E(U,U)\,dx, \end{aligned} \tag{3.16}$$

where $E(U,U)$ has the form (3.12), and

$$\begin{aligned} U\cdot P(\partial z,n)U &= (\lambda+2\mu)(n\cdot u)\,div\,u + (\varepsilon+2\nu)(n\cdot\omega)\,div\omega - \\ &\quad -(\mu+\alpha)u\cdot(n\times rot\,u) - (\nu+\beta)\omega\cdot(n\times rot\omega) + \\ &\quad +2\alpha u\cdot(n\times\omega). \end{aligned} \tag{3.17}$$

By the boundary conditions of problems $(M\cdot III)_0^\pm$ and $(M\cdot IV)_0^\pm$, from (3.17) we obtain

$$[U(z)]^\pm\cdot[P(\partial z,n)U(z)]^\pm = 0,\quad z\in\partial\Omega. \tag{3.18}$$

Since $M(\partial x)U(x) = 0$, $x\in\Omega^\pm$, taking into account (3.17), from (3.16) we derive

$$\int\limits_{\Omega^\pm} E(U,U)\,dx = 0.$$

By virtue of (2.1), from (3.12) we obtain $E(U,U)\geq 0$. Then the latter equality implies $E(U,U) = 0$, $x\in\Omega^\pm$, i.e.

$$\begin{aligned} &div\,u(x) = 0,\quad div\omega(x) = 0,\quad rot\,u(x) = 0,\quad rot\omega(x) = 0, \\ &\qquad\qquad rot\,u(x) - 2\omega(x) = 0,\quad x\in\Omega^\pm. \end{aligned}$$

Hence it follows that $\omega(x) = 0$, $x \in \Omega^{\pm}$, $u(x)$ is a harmonic vector that can be represented as

$$u(x) = grad\,\Phi(x), \quad \Delta\Phi(x) = 0, \quad x \in \Omega^{\pm}. \tag{3.19}$$

In the case of problem $(M \cdot IV)_0^{\pm}$ we have $\Delta u(x) = 0$, $x \in \Omega^{\pm}$, and $[u(z)]^{\pm} = 0$, $z \in \partial\Omega$. This is the homogeneous Dirichlet problem which has only a trivial solution, i.e. $u(x) = 0$, $x \in \Omega^{\pm}$.

Taking into account (3.19), from the boundary conditions $[n(z) \cdot u(z)]^{\pm} = 0$, $z \in \partial\Omega$, we obtain

$$\Delta\Phi(x) = 0, \quad x \in \Omega^{\pm} \quad and \quad \left[\frac{\partial\Phi(z)}{\partial n(z)}\right]^{\pm} = 0, \quad z \in \partial\Omega.$$

This is the homogeneous Neumann problem which has a solution $\Phi(x) = C = const$, $x \in \Omega^{\pm}$. Using this fact in (3.19), we obtain $u(x) = 0$, $x \in \Omega^{\pm}$.

Thus problems $(M \cdot III)^{\pm}$ and $(M \cdot IV)^{\pm}$ have only a trivial solution.

Theorem 13 *If $\partial\Omega \in \mathcal{L}_1(\alpha)$, $0 < \alpha \le 1$, then problems $(\overset{\sigma}{M} \cdot III)^{-}$ and $(\overset{\sigma}{M} \cdot IV)^{-}$ admit at most one regular solution.*

Proof. The theorem will be proved, if we show that homogeneous problems $(\overset{\sigma}{M} \cdot III)_0^{-}$, $(\overset{\sigma}{M} \cdot IV)_0^{-}$ ($f_4 = 0$, $f^{(j)} = 0$, $j = 1, 2$) have only a trivial solution.

Denote by $B(0, R)$ the ball bounded by the spherical surface $S(0, R)$ centered at the origin and having radius R for which $\partial\Omega \subset B(0, R)$. Let $\Omega_R^{-} = \Omega^{-} \cap B(0, R)$.

Write the Green formula (3.15) for the domain Ω_R^{-},

$$\begin{aligned} &\int\limits_{\Omega_R^-} \Big[U(x) \cdot M(\partial x)\overline{U}(x) + E(U, \overline{U})\Big]\, dx = \\ = -&\int\limits_{\partial\Omega} [U(z)]^{-} \cdot \Big[P(\partial z, n)\overline{U}(z)\Big]^{-}\, ds + \\ + &\int\limits_{S(0,R)} U(z) \cdot P(\partial z, n_0)\overline{U}(z)\, ds, \end{aligned} \tag{3.20}$$

where U and $\overline{U}$ are the complex-conjugate vectors, $n_0(z)$ is the unit vector of the outward normal with respect to Ω_R^{-} at a point $z \in S(0, R)$.

Since $E(U, \overline{U}) = E(\overline{U}, U)$, (3.20) implies

$$\int\limits_{\Omega_R^-} \Big[U(x) \cdot M(\partial x)\overline{U}(x) - \overline{U}(x) \cdot M(\partial x)U(x)\Big]\, dx =$$

$$= - \int_{\partial\Omega} \Big[U(z) \cdot P(\partial z, n)\overline{U}(z) - \overline{U}(z) \cdot P(\partial z, n)U(z)\Big]^{-} +$$
$$+ \int_{S(0,R)} \Big[U(z) \cdot P(\partial z, n_0)\overline{U}(z) - \overline{U}(z) \cdot P(\partial z, n_0)U(z)\Big] \, ds.$$

If in this equality we take into account the boundary conditions of problems $(\overset{\sigma}{M} \cdot III)_0^-$ and $(\overset{\sigma}{M} \cdot IV)_0^-$ and also equality (3.8), then we obtain

$$Im \int_{S(0,R)} \overline{U}(z) \cdot P(\partial z, n_0)\, U(z) \, ds = 0, \tag{3.21}$$

where

$$\begin{aligned} \overline{U} \cdot P(\partial z, n_0)U &= (\lambda + 2\mu)(n_0 \cdot \overline{u})\, div\, u + (\varepsilon + 2\nu)(n_0 \cdot \overline{\omega})\, div \omega - \\ &\quad -(\mu + \alpha)\overline{u} \cdot (n_0 \times rot\, u) - (\nu + \beta)\overline{\omega} \cdot (n_0 \times rot \omega) + \\ &\quad +2\alpha\overline{u} \cdot (n_0 \times \omega). \end{aligned} \tag{3.22}$$

By virtue of Remark 2.8, a solution of system (2.13) is represented as (2.27).

The following estimates are true [9]:

$$\begin{aligned} div\, V^{(j)}(x) &= ik_j \Big(n_0 \cdot V^{(j)}(x)\Big) + O(R^{-2}), \;\; j = 1, 2, \\ rot\, V^{(j)}(x) &= ik_j \Big(n_0 \times V^{(j)}(x)\Big) + O(R^{-2}), \;\; j = 3, 4, \\ n_0 \cdot V^{(j)}(x) &= O(R^{-2}), \;\; j = 3, 4, \\ n_0 \times V^{(j)}(x) &= O(R^{-2}), \;\; j = 1, 2, \\ \overline{V}^{(\ell)}(x) \cdot V^{(j)}(x) &= O(R^{-3}), \;\; \ell = 1, 2, \;\; j = 3, 4. \end{aligned} \tag{3.23}$$

These estimates yield

$$\begin{aligned} \overline{V}^{(\ell)}(x) \cdot \Big(n_0 \times V^{(j)}(x)\Big) &= O(R^{-3}), \;\; \ell = 1, 2, \;\; j = 3, 4, \\ \overline{V}^{(\ell)}(x) \cdot \Big[n_0 \times \Big(n_0 \times V^{(j)}(x)\Big)\Big] &= O(R^{-3}), \;\; \ell = 1, 2, \;\; j = 3, 4, \\ \overline{V}^{(\ell)}(x) \cdot \Big[n_0 \times \Big(n_0 \times V^{(j)}(x)\Big)\Big] &= -\overline{V}^{(\ell)}(x) \cdot V^{(j)}(x) + O(R^{-3}), \\ j, \ell &= 3, 4, \\ \Big(n_0 \times \overline{V}^{(\ell)}(x)\Big) \cdot \Big(n_0 \times V^{(j)}(x)\Big) &= \overline{V}^{(\ell)}(x) \cdot V^{(j)}(x) + O(R^{-3}), \\ j, \ell &= 3, 4. \end{aligned} \tag{3.24}$$

Taking into account estimates (3.23), (3.24) and equalities (2.27), we obtain

$$(n_0 \cdot \overline{u})\, div\, u = ik_1 \Big| n_0 \cdot V^{(1)} \Big|^2 + O(R^{-3}),$$

$$\begin{aligned}
(n_0 \cdot \overline{\omega}) div\omega &= ik_2 \left| n_0 \cdot V^{(2)} \right|^2 + O(R^{-3}), \\
\overline{u} \cdot (n_0 \times rot u) &= -ik_3\alpha_3^2 |V^{(3)}|^2 - ik_4\alpha_4^2 |V^{(4)}|^2 - \\
&\quad -i\alpha_3\alpha_4 \left[k_4 \overline{V}^{(3)} \cdot V^{(4)} + k_3 \overline{V}^{(4)} \cdot V^{(3)} \right] + \\
&\quad +O(R^{-3}), \\
\overline{\omega} \cdot (n_0 \times rot\omega) &= -ik_3^3\beta_3^2 |V^{(3)}|^2 - ik_4^3\beta_4^2 |V^{(4)}|^2 - \qquad (3.25) \\
&\quad -ik_3k_4\beta_3\beta_4 \left[k_4 \overline{V}^{(3)} \cdot V^{(4)} + k_3 \overline{V}^{(4)} \cdot V^{(3)} \right] + \\
&\quad +O(R^{-3}), \\
\overline{u} \cdot (n_0 \times \omega) &= -ik_3\alpha_3\beta_3 |V^{(3)}|^2 - ik_4\alpha_4\beta_4 |V^{(4)}|^2 - \\
&\quad -ik_3\alpha_4\beta_3 V^{(3)} \cdot \overline{V}^{(4)} - ik_4\alpha_3\beta_4 \overline{V}^{(3)} \cdot V^{(4)} + \\
&\quad +O(R^{-3}).
\end{aligned}$$

Using these estimates in (3.22) we obtain

$$\begin{aligned}
\overline{U} \cdot P(\partial z, n_0) U &= ik_1(\lambda + 2\mu) \left| n_0 \cdot V^{(1)} \right|^2 + \\
&\quad +ik_2(\varepsilon + 2\nu) \left| n_0 \cdot V^{(2)} \right|^2 + i\gamma_1 |V^{(3)}|^2 + i\gamma_2 |V^{(4)}|^2 \\
&\quad +i\gamma_3 \left(\overline{V}^{(3)} \cdot V^{(4)} - V^{(3)} \cdot \overline{V}^{(4)} \right) + O(R^{-3}), \quad (3.26)
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= (\mu + \alpha) k_3^3 \left[(\mu + \alpha)(\nu + \beta)(k_3^2 - \sigma_1^2)^2 + 4\alpha^2 \sigma_1^2 \right] > 0, \\
\gamma_2 &= (\mu + \alpha) k_4^3 \left[(\mu + \alpha)(\nu + \beta)(k_3^2 - \sigma_1^2)^2 + 4\alpha^2 \sigma_1^2 \right] > 0, \\
\gamma_3 &= 4\alpha^2 (\mu + \alpha) k_3 k_4 (k_3 - k_4) \sigma_1^2.
\end{aligned}$$

From (3.26) it follows that

$$\begin{aligned}
Im \overline{U}(z) \cdot P(\partial z, n_0) U(z) &= k_1(\lambda + 2\mu) \left| n_0(z) \cdot V^{(1)}(z) \right|^2 \\
&\quad +k_2(\varepsilon + 2\nu) \left| n_0(z) \cdot V^{(2)}(z) \right|^2 \\
&\quad +\gamma_1 |V^{(3)}(z)|^2 + \gamma_2 |V^{(4)}(z)|^2 + O(R^{-3}).
\end{aligned}$$

Using this equality in (3.21) we have

$$\begin{aligned}
\lim_{R \to \infty} \int\limits_{S(0,R)} \left| n_0(z) \cdot V^{(j)}(z) \right|^2 ds &= 0, \ \ j = 1, 2, \\
\lim_{R \to \infty} \int\limits_{S(0,R)} |V^{(j)}(z)|^2 ds &= 0, \ \ j = 3, 4. \qquad (3.27)
\end{aligned}$$

The following lemma is valid [9].

Lemma 14 *A regular in* Ω^- *solution of the equation* $(\Delta+\lambda^2)u = 0$, $\lambda^2 > 0$, *satisfying the radiation condition and the condition*

$$\lim_{R\to\infty} \int\limits_{S(0,R)} |u(z)|^2\, ds = 0$$

is identical zero.

The following lemma has been proved (see in this journal the paper by L. Giorgashvili and K. Skhvitaridze).

Lemma 15 *A regular in* Ω^- *solution of the equation* $(\Delta + \lambda^2)u = 0$, $rot\, u = 0$, $\lambda^2 > 0$, *satisfying the radiation condition and the condition*

$$\lim_{R\to\infty} \int\limits_{S(0,R)} |n_0(z)\cdot u(z)|^2\, ds = 0$$

is identical zero.

By these lemmas, (3.27) implies $V^{(j)}(x) = 0$, $j = 1,2,3,4$. Hence, by representations (2.27), we obtain $u(x) = 0$, $\omega(x) = 0$. This means that problems $(\overset{\sigma}{M}\cdot III)^-$ and $(\overset{\sigma}{M}\cdot IV)^-$ admit at most one regular solution.

Theorem 16 *If* $\partial\Omega \in \mathcal{L}_1(\alpha)$, $0 < \alpha \le 1$, *then problems* $(\overset{\sigma}{M}\cdot I)^-$ *and* $(\overset{\sigma}{M}\cdot II)^-$ *admit at most one regular solution.*

Proof. Let us write the Green formula in the domain Ω_R^- [9]

$$\begin{aligned}
&\int\limits_{\Omega_R^-} \Big[U(x)\cdot M(\partial x)\overline{U}(x) - \overline{U}(x)\cdot M(\partial x)U(x)\Big]\, dx \\
&= \int\limits_{\partial\Omega} \Big[U(z)\cdot T(\partial z,n)\overline{U}(z) - \overline{U}(z)\cdot T(\partial z,n)U(z)\Big]^-\, ds \\
&\quad + \int\limits_{S(0,R)} \Big[U(z)\cdot T(\partial z,n_0)\overline{U}(z) - \overline{U}(z)\cdot T(\partial z,n_0)U(z)\Big]\, ds,
\end{aligned}$$

where the operator $T(\partial z, n)$ has form (1.5).

Taking into account the boundary conditions of problems $(\overset{\sigma}{M}\cdot I)_0^-$ and $(\overset{\sigma}{M}\cdot II)_0^-$ ($f(z) = 0$, $z \in \partial\Omega$), and also equality (3.8), we obtain

$$Im \int\limits_{S(0,R)} \overline{U}(z)\cdot T(\partial z, n_0)\, U(z)\, ds = 0, \tag{3.28}$$

where

$$\begin{aligned}\overline{U}\cdot T(\partial z,n_0)U &= 2\mu\overline{u}\cdot\tfrac{\partial u}{\partial R}+2\nu\overline{\omega}\cdot\tfrac{\partial\omega}{\partial R}+\lambda(n_0\cdot\overline{u})divu\\ &+\varepsilon(n_0\cdot\overline{\omega})div\omega+(\mu-\alpha)\overline{u}\cdot[n_0\times rotu]\\ &+(\nu-\beta)\overline{\omega}\cdot[n_0\times rot\omega]+2\alpha\overline{u}\cdot[n_0\times\omega].\end{aligned}$$

Using estimates (2.28) and (3.25) we have

$$\begin{aligned}\overline{U}\cdot T(\partial z,n_0)U &= i(\lambda+2\mu)k_1|n_0\cdot V^{(1)}|^2\\ &+ik_2(\varepsilon+2\nu)|n_0\cdot V^{(2)}|^2+i2\mu k_1\left(|V^{(1)}|^2-|n_0\cdot V^{(1)}|^2\right)\\ &+i2\nu k_2\left(|V^{(2)}|^2-|n_0\cdot V^{(2)}|^2\right)+i\gamma_1|V^{(3)}|^2+i\gamma_2|V^{(4)}|^2\\ &+i\gamma_3\left(\overline{V}^{(3)}\cdot V^{(4)}-V^{(3)}\cdot\overline{V}^{(4)}\right)+O(R^{-3}),\end{aligned}\tag{3.29}$$

where γ_j, $j=1,2,3$, are the constants from (3.26).

(3.29) implies

$$\begin{aligned}&Im\overline{U}\cdot T(\partial z,n_0)U\\ =\ &(\lambda+2\mu)k_1|n_0\cdot V^{(1)}|^2+(\varepsilon+2\nu)k_2|n_0\cdot V^{(2)}|^2\\ &+2\mu k_1\left(|V^{(1)}|^2-|n_0\cdot V^{(1)}|^2\right)+2\nu k_2\left(|V^{(2)}|^2-|n_0\cdot V^{(2)}|^2\right)\\ &+\gamma_1|V^{(3)}|^2+\gamma_2|V^{(4)}|^2+O(R^{-3}).\end{aligned}$$

Hence, by virtue of (3.28), we obtain

$$\begin{aligned}\lim_{R\to\infty}\int\limits_{S(0,R)}\left|n_0(z)\cdot V^{(j)}(z)\right|^2ds &=0,\ \ j=1,2,\\ \lim_{R\to\infty}\int\limits_{S(0,R)}\left(|V^{(j)}(z)|^2-\left|n_0(z)\cdot V^{(j)}(z)\right|^2\right)ds &=0,\ \ j=1,2,\\ \lim_{R\to\infty}\int\limits_{S(0,R)}|V^{(j)}(z)|^2\,ds &=0,\ \ j=3,4.\end{aligned}$$

These equalities imply

$$\lim_{R\to\infty}\int\limits_{S(0,R)}|V^{(j)}(z)|^2\,ds=0,\ \ j=1,2,3,4,$$

from which, by Lemma 3.5, we obtain $V^{(j)}(x)=0$, $j=1,2,3,4$, $x\in\Omega^-$. The substitution of the values of the vectors $V^{(j)}(x)$, $j=1,2,3,4$, into (2.27) gives $u(x)=0$, $\omega(x)=0$, $x\in\Omega^-$.

Thus the homogeneous problems $(\overset{\sigma}{M}\cdot I)_0^-$ and $(\overset{\sigma}{M}\cdot II)_0^-$ have only a trivial solution. This means that the nonhomogeneous problems admit at most one regular solution.

Theorem 3.7 is proved by a different technique in [9].

4. Solution of the Static Problems

Let Ω^+ be the ball bounded by the spherical surface $\partial\Omega$ with center at the origin and radius R, $\Omega^- = R^3 \setminus \overline{\Omega}^+$.

A solution of the internal problems is to be sought in form (2.2), where [15]

$$\begin{aligned}
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\left(\frac{r}{R}\right)^k Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}, \quad j=1,2,3,\\
\Phi_4(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(\lambda_1 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(4)}, \qquad (4.1)\\
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(\lambda_2 r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)}, \quad j=5,6,
\end{aligned}$$

here (r,ϑ,φ) are the spherical coordinates of a point $x\in\Omega^+$, $A_{mk}^{(j)}$, $j = 1,2,\dots,6$ are the sought constants,

$$\begin{aligned}
Y_k^{(m)}(\vartheta,\varphi) &= \sqrt{\frac{2k+1}{4\Pi}\cdot\frac{(k-m)!}{k+m)!}}\,P_k^{(m)}(\cos\vartheta)e^{im\varphi},\\
g_k(\lambda_j r) &= \sqrt{\frac{R}{r}}\,\frac{I_{k+1/2}(\lambda_j r)}{I_{k+1/2}(\lambda_j R)}, \quad j=1,2,
\end{aligned}$$

$P_k^{(m)}(\cos\vartheta)$ is the adjoint Legendre function, $I_{k+1/2}(x)$ is the Bessel function of the imaginary argument and half-integral order.

Let us assume that the functions $\Phi_j(x)$, $j=1,3,5,6$, satisfy the condition

$$\int\limits_{\partial\Omega_1}\Phi_j(x)\,ds = 0, \quad j=1,3,5,6, \qquad (4.2)$$

where $\partial\Omega_1 = \{x:\ x\in R^3,\ |x| = R_1\}$, $0<R_1<R$.

If the values of the function $\Phi_j(x)$, $j=1,3,5,6$, from (4.1) are substituted into (4.2), then we have $A_{00}^{(j)} = 0$, $j=1,3,5,6$.

Substituting the values of the function $\Phi_j(x)$, $j=1,2,\dots,6$, from (4.1) into (2.2) and taking into account the equalities

$$\begin{aligned}
grad\left[a(r)\,Y_k^{(m)}(\vartheta,\varphi)\right] &= \frac{da(r)}{dr}X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\,\frac{a(r)}{r}Y_{mk}(\vartheta,\varphi),\\
rot\left[xa(r)\,Y_k^{(m)}(\vartheta,\varphi)\right] &= \sqrt{k(k+1)}\,a(r)Z_{mk}(\vartheta,\varphi), \qquad (4.3)
\end{aligned}$$

$$\begin{aligned} rot rot\left[xa(r)\,Y_k^{(m)}(\vartheta,\varphi)\right] &= \frac{k(k+1)}{r}\,a(r)X_{mk}(\vartheta,\varphi)+\\ &+\sqrt{k(k+1)}\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Y_{mk}(\vartheta,\varphi), \end{aligned}$$

we obtain

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\left\{u_{mk}^{(1)}(r)X_{mk}(\vartheta,\varphi)+\right.\\ &\left.+\sqrt{k(k+1)}\left[v_{mk}^{(1)}(r)Y_{mk}(\vartheta,\varphi)+\omega_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)\right]\right\},\\ \omega(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\left\{u_{mk}^{(2)}(r)X_{mk}(\vartheta,\varphi)+\right. \qquad (4.4)\\ &\left.+\sqrt{k(k+1)}\left[v_{mk}^{(2)}(r)Y_{mk}(\vartheta,\varphi)+\omega_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)\right]\right\}, \end{aligned}$$

where

$$\begin{aligned} u_{mk}^{(1)}(r) &= \frac{k}{R}\left(\frac{r}{R}\right)^{k-1}A_{mk}^{(1)}+R(k+1)(bk-2a)\left(\frac{r}{R}\right)^{k+1}A_{mk}^{(2)}\\ &+ \frac{2\alpha k(k+1)}{r}\,g_k(\lambda_2 r)A_{mk}^{(5)},\quad k\geq 0,\\ v_{mk}^{(1)}(r) &= \frac{1}{R}\left(\frac{r}{R}\right)^{k-1}A_{mk}^{(1)}+R(bk+b+2)\left(\frac{r}{R}\right)^{k+1}A_{mk}^{(2)}\\ &+ 2\alpha\left(\frac{d}{dr}+\frac{1}{r}\right)g_k(\lambda_2 r)A_{mk}^{(5)},\quad k\geq 1,\\ \omega_{mk}^{(1)}(r) &= \left(\frac{r}{R}\right)^{k}A_{mk}^{(3)}+2\alpha g_k(\lambda_2 r)A_{mk}^{(6)},\quad k\geq 1,\\ u_{mk}^{(2)}(r) &= \frac{k(k+1)}{2R}\left(\frac{r}{R}\right)^{k-1}A_{mk}^{(3)}+\frac{d}{dr}\,g_k(\lambda_1 r)A_{mk}^{(4)}\\ &+ \frac{(\mu+\alpha)k(k+1)}{r}\,g_k(\lambda_2 r)A_{mk}^{(6)},\quad k\geq 0,\\ v_{mk}^{(2)}(r) &= \frac{k+1}{2R}\left(\frac{r}{R}\right)^{k-1}A_{mk}^{(3)}+\frac{1}{r}\,g_k(\lambda_1 r)A_{mk}^{(4)}\\ &+ (\mu+\alpha)\left(\frac{d}{dr}+\frac{1}{r}\right)g_k(\lambda_2 r)A_{mk}^{(6)},\quad k\geq 1,\\ \omega_{mk}^{(2)}(r) &= -(2k+3)\left(\frac{r}{R}\right)^{k}A_{mk}^{(2)}-(\mu+\alpha)g_k(\lambda_2 r)A_{mk}^{(5)},\quad k\geq 1, \end{aligned}$$

here $b=(\lambda+\mu)(\lambda+2\mu)^{-1}$,

$$X_{mk}(\vartheta,\varphi) = e_r Y_k^{(m)}(\vartheta,\varphi),\quad k\geq 0,$$

$$
\begin{aligned}
Y_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}}\left(e_\vartheta\frac{\partial}{\partial\vartheta}+\frac{e_\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)Y_k^{(m)}(\vartheta,\varphi),\quad k\ge 1,\\
Z_{mk}(\vartheta,\varphi) &= \frac{1}{\sqrt{k(k+1)}}\left(\frac{e_\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi}-e_\varphi\frac{\partial}{\partial\vartheta}\right)Y_k^{(m)}(\vartheta,\varphi),\quad k\ge 1,
\end{aligned}
$$

$|m|\le k$, e_r, e_ϑ, e_φ are the unit orthogonal vectors

$$
\begin{aligned}
e_r &= (\cos\varphi\sin\vartheta,\sin\varphi\sin\vartheta,\cos\vartheta)^\top,\\
e_\vartheta &= (\cos\varphi\cos\vartheta,\sin\varphi\cos\vartheta,-\sin\vartheta)^\top,\qquad e_\varphi=(-\sin\varphi,\cos\varphi,0)^\top.
\end{aligned}
$$

In the sequel, in formula (4.4) and analogous series the summation index k of the terms, which contain $Y_{mk}(\vartheta,\varphi)$ and $Z_{mk}(\vartheta,\varphi)$, changes from 1 to $+\infty$.

We rewrite the boundary condition of problem $(M\cdot II)^+$ as

$$
\begin{aligned}
\Big[T^{(1)}(\partial z,n)u(z)+T^{(2)}(\partial z,n)\omega(z)\Big]^+ &= f^{(1)}(z),\quad z\in\partial\Omega,\\
\Big[T^{(4)}(\partial z,n)\omega(z)\Big]^+ &= f^{(2)}(z),\quad z\in\partial\Omega, \qquad (4.5)
\end{aligned}
$$

where

$$
\begin{aligned}
T^{(1)}(\partial x,n)u(x)+T^{(2)}(\partial x,n)\omega(x) &= 2\mu\frac{\partial u(x)}{\partial n(x)}+\lambda n\, div u(x)\\
&\quad+(\mu-\alpha)\left[n\times rot u(x)\right]+2\alpha(n\times\omega(x)), \qquad (4.6)\\
T^{(4)}(\partial x,n)\omega(x) &= 2\nu\frac{\partial\omega(x)}{\partial n(x)}+\varepsilon n\, div\omega(x)+(\nu-\beta)\left[n\times rot\omega(x)\right].
\end{aligned}
$$

Substituting the value of the vector $U=(u,\omega)^\top$ from (4.4) into (4.6) and taking into account the identities [8]

$$
\begin{aligned}
e_r\cdot X_{mk}(\vartheta,\varphi) &= Y_k^{(m)}(\vartheta,\varphi),\quad e_r\cdot Y_{mk}(\vartheta,\varphi)=0,\quad e_r\cdot Z_{mk}(\vartheta,\varphi)=0,\\
e_r\times X_{mk}(\vartheta,\varphi) &= 0,\quad e_r\times Y_{mk}(\vartheta,\varphi)=-Z_{mk}(\vartheta,\varphi),\\
e_r\times Z_{mk}(\vartheta,\varphi) &= Y_{mk}(\vartheta,\varphi),\\
div\left[a(r)X_{mk}(\vartheta,\varphi)\right] &= \left(\frac{d}{dr}+\frac{2}{r}\right)a(r)\,Y_k^{(m)}(\vartheta,\varphi),\\
div\left[a(r)Y_{mk}(\vartheta,\varphi)\right] &= -\sqrt{k(k+1)}\,\frac{a(r)}{r}\,Y_k^{(m)}(\vartheta,\varphi),\\
div\left[a(r)Z_{mk}(\vartheta,\varphi)\right] &= 0, \qquad (4.7)\\
rot\left[a(r)X_{mk}(\vartheta,\varphi)\right] &= \sqrt{k(k+1)}\,\frac{a(r)}{r}\,Z_{mk}(\vartheta,\varphi),\\
rot\left[a(r)Y_{mk}(\vartheta,\varphi)\right] &= -\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Z_{mk}(\vartheta,\varphi),
\end{aligned}
$$

$$
\begin{aligned}
rot\,[a(r)Z_{mk}(\vartheta,\varphi)] \;=\;& \sqrt{k(k+1)}\,\frac{a(r)}{r}\,X_{mk}(\vartheta,\varphi)+\\
&+\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)Y_{mk}(\vartheta,\varphi),
\end{aligned}
$$

we obtain

$$
\begin{gathered}
T^{(1)}(\partial x,n)u(x)+T^{(2)}(\partial x,n)\omega(x)=\\
=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{a^{(1)}_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
+\sqrt{k(k+1)}\,\Big[b^{(1)}_{mk}(r)Y_{mk}(\vartheta,\varphi)+c^{(1)}_{mk}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},
\end{gathered}
\tag{4.8}
$$

$$
\begin{gathered}
T^{(4)}(\partial x,n)\omega(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{a^{(2)}_{mk}(r)X_{mk}(\vartheta,\varphi)+\\
+\sqrt{k(k+1)}\,\Big[b^{(2)}_{mk}(r)Y_{mk}(\vartheta,\varphi)+c^{(2)}_{mk}(r)Z_{mk}(\vartheta,\varphi)\Big]\Big\},
\end{gathered}
$$

where

$$
\begin{aligned}
a^{(1)}_{mk}(r)=&\frac{2\mu k(k-1)}{R^2}\left(\frac{r}{R}\right)^{k-2}A^{(1)}_{mk}+\\
&+2\mu(k+1)\left[bk(k-1)+1-4b\right]\left(\frac{r}{R}\right)^{k}A^{(2)}_{mk}+\\
&+4\alpha\mu k(k+1)\frac{d}{dr}\left(\frac{1}{r}\,g_k(\lambda_2 r)\right)A^{(5)}_{mk},\quad k\geq\ 0,\\
b^{(1)}_{mk}(r)=&\frac{2\mu k(k-1)}{R^2}\left(\frac{r}{R}\right)^{k-2}A^{(1)}_{mk}+2\mu\left[b(k+1)^2-1\right]\left(\frac{r}{R}\right)^{k}A^{(2)}_{mk}-\\
&-\frac{4\alpha\mu}{r}\left(\frac{d}{dr}-\frac{k^2+k-1}{r}\right)g_k(\lambda_2 r)A^{(5)}_{mk},\quad k\geq\ 1,\\
c^{(1)}_{mk}(r)=&\frac{\mu(k-1)}{R^2}\left(\frac{r}{R}\right)^{k-1}A^{(3)}_{mk}-\frac{2\alpha}{r}\,g_k(\lambda_1 r)A^{(4)}_{mk}-\\
&-\frac{4\alpha\mu}{r}\,g_k(\lambda_2 r)A^{(6)}_{mk},\quad k\geq\ 1,\\
a^{(2)}_{mk}(r)=&\frac{\nu k(k^2-1)}{R^2}\left(\frac{r}{R}\right)^{k-2}A^{(3)}_{mk}+\left(2\nu\,\frac{d^2}{dr^2}+\varepsilon\lambda_1^2\right)g_k(\lambda_1 r)A^{(4)}_{mk}+\\
&+2\nu(\mu+\alpha)k(k+1)\frac{d}{dr}\left(\frac{1}{r}\,g_k(\lambda_2 r)\right)A^{(6)}_{mk},\quad k\geq\ 0,\\
b^{(2)}_{mk}(r)=&\frac{\nu(k^2-1)}{R^2}\left(\frac{r}{R}\right)^{k-2}A^{(3)}_{mk}+2\nu\,\frac{d}{dr}\left(\frac{1}{r}\,g_k(\lambda_1 r)\right)A^{(4)}_{mk}-\\
&-(\mu+\alpha)\left[\frac{2\nu}{r}\left(\frac{d}{dr}-\frac{k^2+k-1}{r}\right)-(\nu+\beta)\lambda_2^2\right]g_k(\lambda_2 r)A^{(6)}_{mk},
\end{aligned}
$$

$$c_{mk}^{(2)}(r) = -\frac{2k+3}{R}\left[(\nu+\beta)(k-1)+2\beta\right]\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(2)} - \\ -(\mu+\alpha)\lambda_2^2\left[(\nu+\beta)\frac{d}{dr}+\frac{\nu-\beta}{r}\right] g_k(\lambda_2 r)A_{mk}^{(5)}, \quad k \geq 1.$$

(4.4) implies

$$\begin{aligned} n(x)\cdot u(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} u_{mk}^{(1)}(r) Y_k^{(m)}(\vartheta,\varphi), \\ n(x)\cdot \omega(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} u_{mk}^{(2)}(r) Y_k^{(m)}(\vartheta,\varphi), \qquad (4.9)\\ n(x)\times rot u(x) &= \\ &= \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}\left[\widetilde{v}_{mk}^{(1)}(r)Y_{mk}(\vartheta,\varphi)+\widetilde{w}_{mk}^{(1)}(r)Z_{mk}(\vartheta,\varphi)\right] \qquad (4.10)\\ n(x)\times rot\omega(x) &= \\ &= \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}\left[\widetilde{v}_{mk}^{(2)}(r)Y_{mk}(\vartheta,\varphi)+\widetilde{w}_{mk}^{(2)}(r)Z_{mk}(\vartheta,\varphi)\right], \end{aligned}$$

where

$$\begin{aligned} \widetilde{v}_{mk}^{(1)}(r) &= -2(2k+3)\left(\frac{r}{R}\right)^{k} A_{mk}^{(2)} - 2\alpha\lambda_2^2 g_k(\lambda_2 r)A_{mk}^{(5)}, \\ \widetilde{w}_{mk}^{(1)}(r) &= -\frac{k+1}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(3)} - 2\alpha\left(\frac{d}{dr}+\frac{1}{r}\right) g_k(\lambda_2 r)A_{mk}^{(6)}, \\ \widetilde{v}_{mk}^{(2)}(r) &= -(\mu+\alpha)\lambda_2^2 g_k(\lambda_2 r)A_{mk}^{(6)}, \\ \widetilde{w}_{mk}^{(2)}(r) &= -\frac{(k+1)(2k+3)}{R}\left(\frac{r}{R}\right)^{k-1} A_{mk}^{(2)} \\ &+ (\mu+\alpha)\lambda_2^2\left(\frac{d}{dr}+\frac{1}{r}\right) g_k(\lambda_2 r)A_{mk}^{(5)}, \quad k\geq 1. \end{aligned}$$

Formulas (4.4), (4.8)–(4.10) allow us to solve all the internal problems $(M)^+$. Let us consider problem $(M\cdot II)^+$ as an example.

Let the vectors $f^{(j)}(z)$, $j=1,2$, satisfy those sufficient conditions, under which they can be expanded in a Fourier-Laplace series in the system $\{X_{mk}(\vartheta,\varphi), Y_{mk}(\vartheta,\varphi), Z_{mk}(\vartheta,\varphi)\}_{|m|\leq k, k=\overline{0,\infty}}$,

$$\begin{aligned} f^{(j)}(z) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{\alpha_{mk}^{(j)} X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\times \\ &\times\left[\beta_{mk}^{(j)} Y_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)} Z_{mk}(\vartheta,\varphi)\right]\Big\}, \qquad (4.11)\\ j &= 1,2, \end{aligned}$$

where $\alpha_{mk}^{(j)}$, $\beta\alpha_{mk}^{(j)}$, $\gamma\alpha_{mk}^{(j)}$, $j=1,2$, are the Fourier coefficients.

If we pass to the limit on both sides of equality (4.8) as $x \to z \in \partial\Omega$, $(r \to R)$ and take into account condition (4.5) and equality (4.11), then, for the unknown constants $A_{mk}^{(j)}$, $j=1,2,\dots,6$, we obtain the following system of algebraic equations

$$
\begin{aligned}
&2\mu(1-4b)A_{00}^{(2)} = \alpha_{00}^{(1)},\\
&-4\left(\frac{\nu}{R}\frac{d}{dR}-\alpha\right) g_0(\lambda_1 R)A_{00}^{(4)} = \alpha_{00}^{(2)},
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
&\frac{k(k-1)}{R^2}\,A_{mk}^{(1)} + (k+1)\left[bk(k-1)+1-4b\right]A_{mk}^{(2)}\\
&+2\alpha k(k+1)\frac{d}{dR}\left(\frac{1}{R}\,g_k(\lambda_2 R)\right)A_{mk}^{(5)} = \frac{1}{2\mu}\,\alpha_{mk}^{(1)},\\
&\frac{k-1}{R^2}\,A_{mk}^{(1)} + \left[b(k-1)^2-1\right]A_{mk}^{(2)} -\\
&-\frac{2\alpha}{R}\left(\frac{d}{dR}-\frac{k^2+k-1}{R}\right) g_k(\lambda_2 R)A_{mk}^{(5)} = \frac{1}{2\mu}\,\beta_{mk}^{(1)},
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
&\frac{2k+3}{R}\,\left[(\nu+\beta)(k-1)+2\beta\right]A_{mk}^{(2)} +\\
&+(\mu+\alpha)\lambda_2^2\left[(\nu+\beta)\frac{d}{dR}+\frac{\nu-\beta}{R}\right] g_k(\lambda_2 R)A_{mk}^{(5)} = -\gamma_{mk}^{(2)},\\
&\frac{\nu k(k^2-1)}{R^2}\,A_{mk}^{(3)} + \left(2\nu\,\frac{d^2}{dR^2}+\varepsilon\lambda_1^2\right) g_k(\lambda_1 R)A_{mk}^{(4)} +\\
&+2\nu(\mu+\alpha)k(k+1)\frac{d}{dR}\left(\frac{1}{R}\,g_k(\lambda_2 R)\right)A_{mk}^{(6)} = \alpha_{mk}^{(2)},\\
&\frac{\nu(k^2-1)}{R^2}\,A_{mk}^{(3)} + 2\nu\,\frac{d}{dR}\left(\frac{1}{R}\,g_k(\lambda_1 R)\right)A_{mk}^{(4)} - (\mu+\alpha)\times\\
&\times\left[\frac{2\nu}{R}\left(\frac{d}{dR}-\frac{k^2+k-1}{R}\right)-(\nu+\beta)\lambda_2^2\right] g_k(\lambda_2 R)A_{mk}^{(6)} = \beta_{mk}^{(2)},\\
&\frac{\mu(k-1)}{R}\,A_{mk}^{(3)} - \frac{2\alpha}{R}\,A_{mk}^{(4)} - \frac{4\alpha\mu}{R}\,A_{mk}^{(6)} = \gamma_{mk}^{(1)}.
\end{aligned}
\tag{4.14}
$$

It is implied here that

$$
\frac{d}{dR}\,g_k(\lambda R) = \frac{d}{dr}\,g_k(\lambda r)\bigg|_{r=R}.
$$

The necessary and sufficient condition for problem $(M\cdot II)^+$ to be solvable is that the principal vector and principal moment of external forces acting on the boundary $\partial\Omega$ be equal to zero, i.e.

$$
\int\limits_{\partial\Omega} f^{(1)}(z)\,ds = 0, \quad \int\limits_{\partial\Omega}\left[f^{(2)}(z) + z\times f^{(1)}(z)\right]ds = 0. \tag{4.15}
$$

Substituting the values of the vectors $f^{(1)}(z)$ and $f^{(2)}(z)$ from (4.11) into (4.15), we obtain

$$\alpha^{(1)}_{m1} + 2\beta^{(1)}_{m1} = 0, \quad \alpha^{(2)}_{m1} + 2\beta^{(2)}_{m1} + 2R\gamma^{(1)}_{m1} = 0, \quad m = 0, \pm 1. \tag{4.16}$$

The following lemma is true [11].

Lemma 17 *If the functions $\Phi_j(x)$, $j = 1, 3, 5, 6$, satisfy condition* (4.2)*, then to each zero value of the solution $U = (u, \omega)^\top$ of system* (2.1) *there corresponds the zero value of the function $\Phi_j(x)$, $j = 1, 3, 5, 6$, and vice versa.*

Systems (4.12), (4.13) and (4.14) are compatible. This fact follows from the uniqueness theorem of problem $(M \cdot II)^+$ and Lemma 4.1. For $k = 1$, the compatibility of systems (4.13) and (4.14) is provided by condition (4.15), only the constants $A^{(1)}_{m1}$ and $A^{(3)}_{m1}$ remain undefined. This is natural because the solution of problem $(M \cdot II)^+$ is defined up to an additive vector of rigid displacement.

Substituting the solution of system (4.12)–(4.14) into (4.4) we obtain the solution of problem $(M \cdot II)^+$. Let us prove the convergence of series (4.4) and (4.8). These series converge at each point $x \in \Omega^+$, since for $k \to +\infty$ we have the following asymptotics for the function $g_k(\lambda_j r)$, $j = 1, 2$:

$$g_k(\lambda_j r) \approx \left(\frac{r}{R}\right)^k, \quad \frac{d}{dr}\, g_k(\lambda_j r) \approx k \left(\frac{r}{R}\right)^k.$$

Let $x \in \partial\Omega$, then series (4.4) and (4.8) converge absolutely and uniformly if we prove the convergence of the following majorizing series:

$$\delta \sum_{k=k_0}^{\infty} \sum_{j=1}^{2} k^{3/2} \left[|\alpha^{(j)}_{mk}| + k \left(|\beta^{(j)}_{mk}| + |\gamma^{(j)}_{mk}| \right) \right], \tag{4.17}$$

where the constant δ does not depend on k.

In deriving (4.17) we have used the estimates [8]

$$\begin{aligned} |X_{mk}(\vartheta, \varphi)| &\leq \sqrt{\tfrac{2k+1}{4\pi}}, \quad k \geq 0, \\ |Y_{mk}(\vartheta, \varphi)| &< \sqrt{\tfrac{2k(k+1)}{2k+1}}, \quad k \geq 1, \\ |Z_{mk}(\vartheta, \varphi)| &< \sqrt{\tfrac{2k(k+1)}{2k+1}}, \quad k \geq 1. \end{aligned}$$

Series (4.17) converges if the Fourier coefficients satisfy the requirement

$$\alpha^{(j)}_{mk} = O(k^{-3}), \quad \beta^{(j)}_{mk} = O(k^{-4}), \quad \gamma^{(j)}_{mk} = O(k^{-4}), \quad j = 1, 2. \tag{4.18}$$

The following theorem is valid [8].

Theorem 18 *If $f^{(j)}(z) \in C^{\ell}(\partial\Omega)$, $j = 1, 2$, then the Fourier coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$ admit the following estimates:*

$$\alpha_{mk}^{(j)} = O(k^{-\ell}), \quad \beta_{mk}^{(j)} = O(k^{-\ell-1}), \quad \gamma_{mk}^{(j)} = O(k^{-\ell-1}), \;\; j = 1, 2.$$

From this theorem it follows that estimates (4.18) take place if the boundary vector functions satisfy the sufficient conditions $f^{(j)}(z) \in C^3(\partial\Omega)$, $j = 1, 2$.

Thus if $f^{(j)}(z) \in C^3(\partial\Omega)$, $j = 1, 2$, then the vector $U = (u, \omega)^\top$ represented by (4.4) is a solution of problem $(M \cdot II)^+$.

5. Solution of the Boundary Value Problems of Stationary Oscillations for the Space R^3 with a Spherical Cavity

Let us consider the external problems. A solution of these problems is to be sought in form (2.14), where [15]

$$\begin{aligned}
\Phi_j(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} h_k(k_j r) Y_k^{(m)}(\vartheta, \varphi) A_{mk}^{(j)}, \;\; j = 1, 2, 3, 4, \\
\Phi_{j+2}(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} h_k(k_j r) Y_k^{(m)}(\vartheta, \varphi) A_{mk}^{(j+2)}, \;\; j = 3, 4,
\end{aligned} \tag{5.1}$$

here $A_{mk}^{(j)}$, $j = 1, 2, \ldots, 6$ are the sought constants,

$$h_k(k_j r) = \sqrt{\frac{R}{r}}\, \frac{H_{k+1/2}^{(1)}(k_j r)}{H_{k+1/2}^{(1)}(k_j R)},$$

$H_{k+1/2}^{(1)}(k_j r)$ is Hankel's function of first kind and half-integral order.

Assume that the function $\Phi_j(x)$, $j = 3, 4, 5, 6$, satisfies

$$\int_{\partial\Omega_1} \Phi_j(x)\, ds = 0, \;\; j = 3, 4, 5, 6, \tag{5.2}$$

where $\partial\Omega_1 = \{x : \; x \in R^3, \; |x| = R_1\}$, $R < R_1 < +\infty$.

Substituting the value of the function $\Phi_j(x)$, $j = 3, 4, 5, 6$, from (5.1) into (5.2) we obtain $A_{00}^{(j)} = 0$, $j = 3, 4, 5, 6$.

If we substitute the value of the function $\Phi_j(x)$, $j = 1,2,\ldots,6$, from (5.1) into (2.14) and take into account identities (4.3), then we have

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(1)}(r,\sigma)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\times \\ &\times\Big[v_{mk}^{(1)}(r,\sigma)Y_{mk}(\vartheta,\varphi)+w_{mk}^{(1)}(r,\sigma)Z_{mk}(\vartheta,\varphi)\Big]\Big\}, \\ \omega(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(2)}(r,\sigma)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\times \\ &\times\Big[v_{mk}^{(2)}(r,\sigma)Y_{mk}(\vartheta,\varphi)+w_{mk}^{(2)}(r,\sigma)Z_{mk}(\vartheta,\varphi)\Big]\Big\}, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} u_{mk}^{(1)}(r,\sigma) &= \frac{d}{dr}\,h_k(k_1r)A_{mk}^{(1)}+ \\ +\sum_{j=3}^{4}\alpha_j k(k+1)\,\frac{1}{r}\,h_k(k_jr)A_{mk}^{(j)},\quad k &\geq 0, \\ v_{mk}^{(1)}(r,\sigma) &= \frac{1}{r}\,h_k(k_1r)A_{mk}^{(1)}+ \\ +\sum_{j=3}^{4}\alpha_j\left(\frac{d}{dr}+\frac{1}{r}\right)h_k(k_jr)A_{mk}^{(j)},\quad k &\geq 1, \\ w_{mk}^{(1)}(r,\sigma) &= \sum_{j=3}^{4}\alpha_j h_k(k_jr)A_{mk}^{(j+2)},\quad k\geq 1, \\ u_{mk}^{(2)}(r,\sigma) &= \frac{d}{dr}\,h_k(k_2r)A_{mk}^{(2)}+ \\ +\sum_{j=3}^{4}\beta_j k(k+1)\,\frac{1}{r}\,h_k(k_jr)A_{mk}^{(j+2)},\quad k &\geq 0, \\ v_{mk}^{(2)}(r,\sigma) &= \frac{1}{r}\,h_k(k_2r)A_{mk}^{(2)}+ \\ +\sum_{j=3}^{4}\beta_j\left(\frac{d}{dr}+\frac{1}{r}\right)h_k(k_jr)A_{mk}^{(j+2)},\quad k &\geq 1, \\ w_{mk}^{(2)}(r,\sigma) &= \sum_{j=3}^{4}\beta_j k_j^2 h_k(k_jr)A_{mk}^{(j)},\quad k\geq 1. \end{aligned}$$

If we substitute the value of the vector $U=(u,\omega)^{\top}$ from (5.3) into (4.6) and take into account identities (4.7), then we have

$$T^{(1)}(\partial x,n)u(x)+T^{(2)}(\partial x,n)\omega(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{a_{mk}^{(1)}(r,\sigma)X_{mk}(\vartheta,\varphi)+$$

$$+\sqrt{k(k+1)}\left[b^{(1)}_{mk}(r,\sigma)Y_{mk}(\vartheta,\varphi)+c^{(1)}_{mk}(r,\sigma)Z_{mk}(\vartheta,\varphi)\right]\Big\}, \tag{5.4}$$

$$T^{(4)}(\partial x,n)\omega(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{a^{(2)}_{mk}(r,\sigma)X_{mk}(\vartheta,\varphi)+$$

$$+\sqrt{k(k+1)}\left[b^{(2)}_{mk}(r,\sigma)Y_{mk}(\vartheta,\varphi)+c^{(2)}_{mk}(r,\sigma)Z_{mk}(\vartheta,\varphi)\right]\Big\},$$

where

$$\begin{aligned}
a^{(1)}_{mk}(r,\sigma) &= \left(2\mu\,\frac{d^2}{dr^2}-\lambda k_1^2\right)h_k(k_1r)A^{(1)}_{mk}+\\
&\quad +2\mu\sum_{j=3}^{4}\alpha_j k(k+1)\,\frac{d}{dr}\,\left(\frac{1}{r}\,h_k(k_jr)\right)A^{(j)}_{mk},\\
b^{(1)}_{mk}(r,\sigma) &= 2\mu\,\frac{d}{dr}\left(\frac{1}{r}\,h_k(k_1r)\right)A^{(1)}_{mk}-\\
&\quad -2\alpha\sum_{j=3}^{4}\left(k_j^2\left[2\mu\left(\frac{1}{r}\,\frac{d}{dr}-\frac{k^2+k-1}{r^2}\right)+\rho\sigma^2\right]h_k(k_jr)\right)A^{(j)}_{mk},\\
c^{(1)}_{mk}(r,\sigma) &= -\frac{2\alpha}{r}\,h_k(k_2r)A^{(2)}_{mk}+\\
&\quad +2\alpha\sum_{j=3}^{4}\left[\rho\sigma^2\left(\frac{d}{dr}+\frac{1}{r}\right)-2\mu k_j^2\,\frac{1}{r}\right]h_k(k_jr)A^{(j+2)}_{mk},\\
a^{(2)}_{mk}(r,\sigma) &= \left(2\nu\,\frac{d^2}{dr^2}-\varepsilon k_2^2\right)h_k(k_2r)A^{(2)}_{mk}+\\
&\quad +2\nu\sum_{j=3}^{4}\beta_j k(k+1)\,\frac{d}{dr}\,\left(\frac{1}{r}\,h_k(k_jr)\right)A^{(j+2)}_{mk},\\
b^{(2)}_{mk}(r,\sigma) &= 2\nu\,\frac{d}{dr}\left(\frac{1}{r}\,h_k(k_2r)A^{(2)}_{mk}\right)-\\
&\quad -\sum_{j=3}^{4}\beta_j\left[2\nu\left(\frac{1}{r}\,\frac{d}{dr}-\frac{k^2+k-1}{r^2}\right)+(\nu+\beta)k_j^2\right]h_k(k_jr)A^{(j+2)}_{mk},\\
c^{(2)}_{mk}(r,\sigma) &= \sum_{j=3}^{4}\beta_k k_j^2\left[(\nu+\beta)\,\frac{d}{dr}-\frac{\nu-\beta}{r}\right]h_k(k_jr)A^{(j)}_{mk}.
\end{aligned}$$

In view of (4.7), formulas (5.3) imply

$$n(x)\cdot u(x) \;=\; \sum_{k=0}^{\infty}\sum_{m=-k}^{k}u^{(1)}_{mk}(r,\sigma)Y^{(m)}_{k}(\vartheta,\varphi),$$

$$n(x)\cdot\omega(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} u_{mk}^{(2)}(r,\sigma)Y_k^{(m)}(\vartheta,\varphi), \tag{5.5}$$

$$\begin{aligned} n(x)\times rot u(x) &= \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}\times \\ &\times\left[\widetilde{v}_{mk}^{(1)}(r,\sigma)Y_{mk}(\vartheta,\varphi)+\widetilde{w}_{mk}^{(1)}(r,\sigma)Z_{mk}(\vartheta,\varphi)\right], \\ n(x)\times rot\omega(x) &= \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\sqrt{k(k+1)}\times \\ &\times\left[\widetilde{v}_{mk}^{(2)}(r,\sigma)Y_{mk}(\vartheta,\varphi)+\widetilde{w}_{mk}^{(2)}(r,\sigma)Z_{mk}(\vartheta,\varphi)\right], \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \widetilde{v}_{mk}^{(1)}(r,\sigma) &= \sum_{j=3}^{4}\alpha_j k_j^2 h_k(k_j r)A_{mk}^{(j)}, \\ \widetilde{w}_{mk}^{(1)}(r,\sigma) &= -\sum_{j=3}^{4}\alpha_j\left(\frac{d}{dr}+\frac{1}{r}\right)h_k(k_j r)A_{mk}^{(j+2)}, \\ \widetilde{v}_{mk}^{(2)}(r,\sigma) &= \sum_{j=3}^{4}\beta_j k_j^2 h_k(k_j r)A_{mk}^{(j+2)}, \\ \widetilde{w}_{mk}^{(2)}(r) &= -\sum_{j=3}^{4}\beta_j\left(\frac{d}{dr}+\frac{1}{r}\right)h_k(k_j r)A_{mk}^{(j)}. \end{aligned}$$

Formulas (5.3)–(5.6) allow us to solve any of the problems $(\overset{\sigma}{M})^-$.

Let us consider problem $(\overset{\sigma}{M}\cdot I)^-$. If we pass to the limit on both sides of equality (5.3) as $x\to z\in\partial\Omega$, and take into account the boundary condition of problem $(\overset{\sigma}{M}\cdot I)^-$ and also formulas (4.11), then, for the unknown constants $A_{mk}^{(j)}$, $j=1,2,\dots,6$, we obtain the following system of algebraic equations:

$$\begin{aligned} \frac{d}{dR}h_0(k_1R)A_{00}^{(1)} &= \alpha_{00}^{(1)}, \\ \frac{d}{dR}h_0(k_2R)A_{00}^{(2)} &= \alpha_{00}^{(2)}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \frac{d}{dR}h_k(k_1R)A_{mk}^{(1)}+\sum_{j=3}^{4}\alpha_j k(k+1)\frac{1}{R}h_k(k_jR)A_{mk}^{(j)} &= \alpha_{mk}^{(1)}, \\ \frac{1}{R}h_k(k_1R)A_{mk}^{(1)}+\sum_{j=3}^{4}\alpha_j\left(\frac{d}{dR}+\frac{1}{R}\right)h_k(k_jR)A_{mk}^{(j)} &= \beta_{mk}^{(1)}, \end{aligned} \tag{5.8}$$

$$\begin{aligned}
\sum_{j=3}^{4} \beta_j k_j^2 h_k(k_j R) A_{mk}^{(j)} &= \gamma_{mk}^{(2)}, \quad k \geq 1, \\
\frac{d}{dR} h_k(k_2 R) A_{mk}^{(2)} + \sum_{j=3}^{4} \beta_j k(k+1) \frac{1}{R} h_k(k_j R) A_{mk}^{(j+2)} &= \alpha_{mk}^{(2)}, \\
\frac{1}{R} h_k(k_2 R) A_{mk}^{(2)} + \sum_{j=3}^{4} \beta_j \left(\frac{d}{dR} + \frac{1}{R} \right) h_k(k_j R) A_{mk}^{(j+2)} &= \beta_{mk}^{(2)}, \qquad (5.9) \\
\sum_{j=3}^{4} \alpha_j k_j^2 h_k(k_j R) A_{mk}^{(j+2)} &= \gamma_{mk}^{(1)}, \quad k \geq 1.
\end{aligned}$$

Let us prove the following statement.

Lemma 19 *If the functions $\Phi_j(x)$, $j = 3,4,5,6$, satisfy condition* (5.2)*then to each zero value of the solution $U = (u, \omega)^\top$ of system* (2.13) *there corresponds the zero value of the function $\Phi_j(x)$, $j = 1, 2, \ldots, 6$.*

Proof. From (2.14) it follows that

$$\begin{aligned}
\Phi_1(x) &= -\frac{1}{k_1^2}\, div u(x), \quad \Phi_2(x) = -\frac{1}{k_2^2}\, div \omega(x), \\
& r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + k_3^2 \right) \Phi_3(x) \\
&= \frac{1}{2\alpha\rho\sigma^2 (k_4^2 - k_3^2) k_3^2} \left[\beta_4 x \cdot rot rot u - \alpha_4 x \cdot rot \omega \right], \\
& r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + k_4^2 \right) \Phi_4(x) \\
&= \frac{1}{2\alpha\rho\sigma^2 (k_4^2 - k_3^2) k_4^2} \left[-\beta_3 x \cdot rot rot u + \alpha_3 x \cdot rot \omega \right], \\
& r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + k_3^2 \right) \Phi_5(x) \\
&= \frac{1}{2\alpha\rho\sigma^2 (k_4^2 - k_3^2)} \left[\beta_4 x \cdot rot u - \alpha_4 \left(x \cdot \omega + \frac{1}{k_2^2}\, r \frac{\partial}{\partial r}\, div \omega \right) \right], \\
& r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + k_4^2 \right) \Phi_6(x) \\
&= \frac{1}{2\alpha\rho\sigma^2 (k_4^2 - k_3^2)} \left[-\beta_3 x \cdot rot u + \alpha_3 \left(x \cdot \omega + \frac{1}{k_2^2}\, r \frac{\partial}{\partial r}\, div \omega \right) \right].
\end{aligned}$$

Assuming in these equalities that $u(x) = 0$ and $\omega(x) = 0$, we obtain

$$\Phi_j(x) = 0, \quad j = 1, 2, \quad x \in \Omega^-, \qquad (5.10)$$

$$\begin{aligned} r^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+k_j^2\right)\Phi_j(x) &= 0, \ \ j=3,4, \ \ x\in\Omega^-, \\ r^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+k_j^2\right)\Phi_{j+2}(x) &= 0, \ \ j=3,4, \ \ x\in\Omega^-. \end{aligned} \tag{5.11}$$

Substituting the values of the function $\Phi_j(x)$, $j=3,4,5,6$, from (5.1) into (5.11) we have

$$\begin{aligned} \sum_{k=0}^{\infty}\sum_{m=-k}^{k} k(k+1)h_k(k_jr)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)} &= 0, \ \ j=3,4, \\ \sum_{k=0}^{\infty}\sum_{m=-k}^{k} k(k+1)h_k(k_jr)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j+2)} &= 0, \ \ j=3,4. \end{aligned}$$

Hence it follows that $A_{mk}^{(j)}=0$, $j=3,4,5,6$, $k\geq 1$. If this value of the constants $A_{mk}^{(j)}$ is substituted into (5.1), then we have

$$\begin{aligned} \Phi_j(x) &= \tfrac{1}{2\sqrt{\pi}}\,h_0(k_jr)A_{00}^{(j)}, \ \ j=3,4, \\ \Phi_j(x) &= \tfrac{1}{2\sqrt{\pi}}\,h_0(k_jr)A_{00}^{(j+2)}, \ \ j=3,4. \end{aligned}$$

Since these functions satisfy condition (5.2), we obtain $A_{00}^{(j)}=0$, $j=3,4,5,6$.

Thus we have established that if $U=(u,\omega)^\top=0$, then $\Phi_j(x)=0$, $j=1,2,\dots,6$. The proof of the second part of the lemma follows from (2.14).

Theorem 3.7 and Lemma 5.1 imply that systems (5.7)–(5.9) are compatible. If the solution of these systems is substituted into (5.3), then we obtain a solution of problem $(\overset{\sigma}{M}\cdot I)^-$. Let us show the convergence of series (5.3) and (5.4).

Since for $k\to\infty$ we have the asymptotics [14]

$$h_k(k_jr)\approx\left(\frac{R}{r}\right)^{k+1}, \quad \frac{d}{dr}\,h_k(k_jr)\approx\frac{k}{r}\left(\frac{R}{r}\right)^{k+1},$$

series (5.3) and (5.4) converge at each point $x\in\Omega^-$.

For $x\in\partial\Omega$ these series coincide absolutely and uniformly if we prove the convergence of the following majorizing series

$$\delta'\sum_{k=k_0}^{\infty}\sum_{j=1}^{2}k^{5/2}\left[|\alpha_{mk}^{(j)}|+k\left(|\beta_{mk}^{(j)}|+|\gamma_{mk}^{(j)}|\right)\right],$$

where $'$ is a positive constant not depending on k.

This series converges if

$$\alpha_{mk}^{(j)} = O(k^{-4}), \quad \beta_{mk}^{(j)} = O(k^{-5}), \quad \gamma_{mk}^{(j)} = O(k^{-5}), \;\; j = 1,2. \qquad (5.12)$$

From Theorem 4.2 it follows that if $f^{(j)}(z) \in C^4(\partial\Omega)$, then estimates (5.12) hold.

Using the asymptotic formulas

$$h_k(k_j r) = e^{ik_j r} O(r^{-1}), \quad \left(\frac{d}{dr} - ik_j\right) h_k(k_j r) = e^{ik_j r} O(r^{-2})$$

as $r \to \infty$, we conclude that the vector $U = (u,\omega)^\top$ defined by formulas (5.3) satisfies the radiation condition.

Thus is $f^{(j)}(z) \in C^4(\partial\Omega)$, $j = 1,2$, then the vector $U = (u,\omega)^\top$ defined by formulas (5.3) is a solution of problem $(\overset{\sigma}{M} \cdot I)$.

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