EXACT AND APPROXIMATE SOLUTIONS OF AN ABSTRACT EQUATION OF THE FIRST ORDER OF HYPERBOLIC TYPE WITH A NON-CONSTANT UNBOUNDED OPERATOR COEFFICIENT IN HILBERT SPACE

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(Received:September 20, 1998)

Abstract

A problem for the first order abstract hyperbolic equation with a non-constant unbounded operator coefficient in Hilbert space is considered. Exact and approximate solutions are constructed. It is shown that the error estimate for the approximate solution has exponential rate of convergence.

Key words and phrases:Hyperbolic type equation; abstract equation; exact and approximate solutions.

AMS subject classification: 35L45; 35A35.

1. *Introduction*

We consider a problem:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} + B(x,t)u(x,t) &= f(x,t), \\ u(x,0) &= \varphi(x), \quad x \in [c,d], \end{aligned} \tag{1}$$

where $u : [c,d] \times [0,T] \to H$ is an unknown vector-valued function in a Hilbert space H, $f : [c,d] \times [0,T] \to H$, $\varphi : [c,d] \to H$ are given vector-valued functions, a is a nonnegative constant, $B(x,t)$ for all fixed (x,t) is a linear, selfadjoint, positive operator with a domain $D(B)$, that does not depend on (x,t) and is densely defined in H.

The case when $H = L_2((c,d) \times (0,T))$ was considered in [1] on the basis of FD-method (functional-difference). Another case $a = 0, B(x,t) \equiv B$ was studied in [2] by means of CT-method (Cayley transform), and on the basis of Fourier-Chebyshev series in [3]. The case $a = 0, B(x,t) \equiv B(t)$ was considered in [4] by means of FD-method and CT-method under the assumption that $B(t)$ is a bounded operator. This work is a generalization of the works [1]-[4]. Here we have obtained a constructive representation of the solution of the problem (1) without assumption about boundedness of the operator $B(x,t)$ (see Theorem 1). On the basis of this representation we have built an approximate solution, whose error estimation has an exponential rate of convergence. The basis of our work is combination of the FD-method and Fourier-Chebyshev series method.

2. *A constructive representation of the problem (1) solution*

By analogy with the ordinary hyperbolic equation we will find the solution of the problem (1) in the determined domain:

$$\bar{\Omega} = \{(x,t) : 0 \leq t \leq T, c \leq x - at \leq d\}.$$

With accounting (1) we can consider its integral analogy:

$$u(x,t) = \varphi(x-at) + \int_0^t [f(x-a(t-s),s) - B(x-a(t-s),s)u(x-a(t-s),s)]ds. \tag{2}$$

Definition 1 *If $u(x,t)$ has strong derivatives from x and t in $\bar{\Omega}$ and it belongs to the domain of the operator definition $B : D(B)$, satisfies the equation and initial condition (1), then $u(x,t)$ is called a strong solution of the problem (1).*

Definition 2 *If $u(x,t) \in D(B)$, when $(x,t) \in \Omega$ and it satisfies the equation (2), then $u(x,t)$ is called weak solution of the problem (1).*

Remark. It's easy to check that if $u(x,t)$ is a strong solution, then it satisfies the equation (2).

We shall find a weak solution of the problem (1).

Let us introduce a grid $\omega_h = \{x_i = ih + c : i = \overline{0, N_1}, h = (d-c)/N_1\}$, on $[c,d]$ and a grid $\omega_\tau = \{t_j = j\tau : j = \overline{0, N_2}, \tau = T/N_2\}$ on $[0,T]$. Let us put through the points $x_i, i = \overline{0, N_1}$ characteristics and through the $t_j, j = \overline{0, N_2}$ straight lines that are parallel to the $x-$axe. A cross of these lines gives us a covering of the domain $\bar{\Omega}$ by a grid:

$$\omega_{h\tau} = \{(x_{i,j}, t_j) : x_{i,j} = x_i + at_j, i = \overline{0, N_1}, j = \overline{0, N_2}.$$

Let us use FD-method for the problem (1) solution. At first this method was proposed in [5] and for the partial cases of the problem (1) it was developed in [1], [2], [4], that was mentioned in the introduction. Therefore we approximate $B(x,t)$ by piecewise-constant operator $B_0(x,t)$ as follows:

$$B_0(x,t) = B(\tilde{x}_i + a\tilde{t}_j, t_j) \quad \forall (x,t) \in \Omega_{i,j},$$

where $(\tilde{x}_i, \tilde{t}_j)$ is a fixed point, that satisfies the following conditions:

$$x_{i-1} \le \tilde{x}_i \le x_i, \quad t_{j-1} \le \tilde{t}_j \le t_j,$$

$$\Omega_{i,j} = \{(x,t) : t_{j-1} \le t \le t_j, x_{i-1} \le x - at \le x_i\}, i = \overline{0, N_1}, j = \overline{0, N_2}.$$

We look for the solution of the problem (2) in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \tag{3}$$

where $u_k(x,t)$ are solutions of the recurrent sequence of problems:

$$\begin{aligned} \frac{\partial u_0(x,t)}{\partial t} + a\frac{\partial u_0(x,t)}{\partial x} + B_0(x,t)u_0(x,t) &= f(x,t), \quad (x,t) \in \Omega, \\ u_0(x,0) &= \varphi(x). \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial u_k(x,t)}{\partial t} + a\frac{\partial u_k(x,t)}{\partial x} + B_0(x,t)u_k(x,t) &= (B_0(x,t) - B(x,t)) \times \\ \times u_{k-1}(x,t), &\quad (x,t) \in \Omega, \\ u_k(x,0) &= 0, \quad k = 1, 2, \ldots . \end{aligned} \tag{5}$$

We can represent a middle solution of the problem (1) (in the sense [6]), if $B(x,t) \equiv B$ is a constant operator, $f(x,t) \equiv 0$, $a = 0$, $\varphi(x) \in H$ in the form [3]:

$$u(x,t) = T(t;B)\varphi(x) = \sum_{k=0}^{\infty} a_k(\Phi_t) T_k^*(B^{-1})\varphi(x), \tag{6}$$

where $T_k^*(\mu) = \cos(k \arccos(2\mu - 1))$ are shifted Chebyshev polynomials of the first type (see i.e. [7]), $k \in N$,

$$a_k = \frac{2}{\pi} \int_0^1 \exp\{-\frac{t}{\mu}\} T_k^*(\mu) \frac{d\mu}{(\mu(\mu-1))^{0.5}}, \qquad k = 1, 2, \ldots,$$

$$a_0 = \frac{1}{\pi} \int_0^1 \exp\{-\frac{t}{\mu}\} T_0^*(\mu) \frac{d\mu}{(\mu(\mu-1))^{0.5}},$$

$T(t;B)$ is a C_0-semigroup with a generator B.

Lemma 1 *Let $B(x,t) \equiv B, B = B^* \geq \lambda_0 I, \lambda_0 \geq 0$, $\overline{D(B)} = H$. If $f(x,t)$, $\varphi(x) \in D(B)$, then for the weak solution of the problem (1) the following constructive representation is held*

$$u(x,t) = T(t;B)\varphi(x-at) + \int_0^t T(t-s;B)f(x-a(t-s),s)ds, \qquad (7)$$

where C_0-semigroup $T(t;B)$ is defined by formula (6).

Proof. Let us substitute representation (7) into the right hand side of (2). Then one can get

$$u(x,t) = \varphi(x-at) + \int_0^t [f(x-a(t-s),s) - BT(s;B)\varphi(x-at) -$$

$$-B\int_0^s T(s-s_1;B)f(x-a(t-s_1),s_1)ds_1]ds.$$

Let us use the equality (see [8]):

$$\frac{dT(s;B)}{ds} + BT(s;B) = 0.$$

It gives us a possibility to perform the above equality to the following form:

$$u(x,t) = \varphi(x-at) + \int_0^t [f(x-a(t-s),s)ds + \int_0^t \frac{d}{ds}T(s;B)ds\varphi(x-at) +$$

$$+\int_0^t \int_0^s \frac{\partial}{\partial s}T(s-s_1;B)f(x-a(t-s_1),s_1)ds_1 ds = T(t;B)\varphi(x-at) +$$

$$+\int_0^t \frac{\partial}{\partial s}\int_0^s T(s-s_1;B)f(x-a(t-s_1),s_1)ds_1 ds = T(t;B)\varphi(x-at) +$$

$$+\int_0^t T(t-s_1;B)f(x-a(t-s_1),s_1)ds_1.$$

This equality confirms that representation (7) is a weak solution of the problem (1). □

Let us look for the solutions of the problems (4),(5) in the domain Ω, by using their solutions in subdomains $\Omega_{i,j}$. We have:

$$u_0^{i,j}(x,t) = T(t-t_{j-1};B_{i,j})u_0^{i,j}(x-a(t-t_{j-1}),t_{j-1}) + \int_{t_{j-i}}^{t} [T(t-s;B_{i,j})\times$$

$$\times f(x-a(t-s),s)ds;$$

$$u_k^{i,j}(x,t) = T(t-t_{j-1};B_{i,j})u_k^{i,j}(x-a(t-t_{j-1}),t_{j-1}) + \int_{t_{j-i}}^{t} T(t-s;B_{i,j})\times$$

$$\times[B_0(x-a(t-s),s)-B(x-a(t-s),s)]u_k^{i,j}(x-a(t-s),s)ds, \quad k=1,2,\ldots.$$

Thus we obtain

$$u_0(x,t) = U(t,0)\varphi(x-at) + \int_0^t U(t,s)f(x-a(t-s),s)ds, \tag{8}$$

$$u_k(x,t) = \int_0^t U(t,s)[B_0(x-a(t-s),s)-B(x-a(t-s),s)]u_{k-1}(x-a(t-s),s)ds,$$

$$k=1,2,\ldots, \tag{9}$$

where $U(t,s)$ is evolution operator, which has a form:

$$U(t,s) = \begin{cases} T(t-s;B_{i,j}), & \text{if } j=p, \\ T(t-t_{j-1};B_{i,j})[\prod_{k=1}^{j-p-1} T(t_{j-k}-t_{j-k-1};B_{i,j-k})]\times & \\ \times T(t_p-s;B_{i,p}), & \text{if } j>p, \end{cases}$$

$$t\in[t_{j-1},t_j], \qquad s\in[t_{p-1},t_p]. \tag{10}$$

If the operator $A: H\to H, \quad A=A^*\geq \lambda I, \lambda>0, \overline{D(A)}=H$ and $(-A)$ is a generator of semigroup $T(t;A)$, then this semigroup is compression, i.e.(see [8]) $||T(t;A)||\leq 1$. So we find from (10) inequality:

$$||U(t,s)||\leq 1. \tag{11}$$

Let us find estimations for $u_0(x,t), \quad u_k(x,t), \quad k=1,2,\dots$. By using (11) we get

$$||u_0(x,t)|| \le ||U(t,0)\varphi(x-at)|| + \int\limits_0^t ||U(t,s)f(x-a(t-s),s)||ds \le$$

$$\le ||\varphi(x-at)|| + \int\limits_0^t ||f(x-a(t-s),s)||ds \le$$

$$\le ||\varphi(x-at)|| + \max_{0\le s\le t} \int\limits_0^s ||f(x-a(s-s_1),s_1)||ds_1 \equiv U_0(x,t)$$

from (8). By analogy with (9) one can get

$$||u_k(x,t)|| \le \int\limits_0^t ||U(t,s)[B_0(x-a(t-s),s)-$$

$$-B(x-a(t-s),s)]u_{k-1}(x-a(t-s),s)||ds \le$$

$$q\int\limits_0^t ||u_{k-1}(x-a(t-s),s)||ds,$$

where

$$q = \sup_{x\in[c,d],0\le s\le t,t\in[0,T]} ||B_0(x-a(t-s),s) - B(x-a(t-s),s)||.$$

It's easy to check by substituting x to $x-a(t-s)$, t to s, that one can transform the above inequality in the following form:

$$||u_k(x-a(t-s),s)|| \le q\int\limits_0^s ||u_{k-1}(x-a(s-s_1),s_1)||ds_1.$$

Thus, by continuation of the chain of inequalities, we can get

$$||u_k(x,t)|| \le q\int\limits_0^t ||u_{k-1}(x-a(t-s_1),s_1)||ds_1 \le$$

$$\le q^2 \int\limits_0^t \int\limits_0^{s_1} ||u_{k-2}(x-a(t-s_2),s_2)||ds_2 \le$$

$$\leq q^k \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} ||u_0(x - a(t - s_k), s_k)|| ds_k \dots ds_1 =$$

$$= q^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} ||u_0(x - a(t-s), s)|| ds \leq$$

$$\leq \frac{(qt)^k}{k!} \max_{0 \leq s \leq t} ||u_0(x - a(t-s), s) = \frac{(qt)^k}{k!} U_0(x, t).$$

Let us use these estimations for the majoration of the series (3)

$$||u(x,t)|| \leq \sum_{k=0}^{\infty} ||u_k(x,t)|| \leq \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} U_0(x,t) = e^{qt} U_0(x,t). \qquad (12)$$

If we cut the series (3) on the $m-$th step, then one can get the following estimation for the approximation

$$||u(x,t) - u^{(m)}(x,t)|| \leq \sum_{k=m+1}^{\infty} ||u_k(x,t)|| \leq \frac{(qt)^{m+1}}{(m+1)!} e^{qt} U_0(x,t). \qquad (13)$$

Theorem 1 *Let $B(x,t) : H \to H$ $\forall (x,t) \in \Omega$, $B(x,t) = B^*(x,t) \geq \lambda_0 I$, $\lambda_0 \geq 0$, $D(B(x,t)) = D(B)$ does not depend on (x,t), $\overline{D(B)} = H$ $f(x,t), \varphi(x) \in D(B)$, $q < \infty$. Then FD-method converges to the weak solution of the problem (1) and estimation (12) is valid.*

Proof. Inequality (12) shows, that the series (3) is convergent in the norm of Hilbert space H. Let us show that its sum is a weak solution of the initial-value problem (1).

Indeed, $u_k(x,t)$, that are defined by (9), are solutions of equations:

$$u_k(x,t) = \int_0^t \{[B_0(x - a(t-s), s) - B(x - a(t-s), s)] u_{k-1}(x - a(t-s), s) -$$

$$-B_0(x - a(t-s), s) u_k(x - a(t-s), s)\} ds \qquad k = 1, 2, \dots \qquad (14)$$

(see Lemma 1), and $u_0(x,t)$, that is defined by equality (8), is a solution of equation:

$$u_0(x,t) = \varphi(x - at) + \int_0^t [f(x - a(t-s), s) - B_0(x - a(t-s), s) u_0(x - a(t-s), s)] ds.$$

Thus, by summing (14) with respect to $k = 0$, to ∞, using convergence of the series (3) in the norm of the space H and dense of the operator $B(x,t)$, one can get a necessary affirmation.

□

3. *Combining FD- and Fourier-Chebyshev series method*

For solving the equations of the problems (4), (5) in $\Omega_{i,j}$ let us use the representation $T(t;B_{i,j})u$ in the form (6), that was obtained in [3].

Let

$$T_N(t;B_{i,j})u=\sum_{k=0}^{N}a_kT_k^*(B_{i,j}^{-1})u,$$

$U_N(t,s)$ be a corresponding operator that one can get from the evolution operator by means of substitution $T(t;B_{i,j})$ for $T_N(t;B_{i,j})$.

Let us estimate the approximate error:

$$z_k(x,t)=u_{k,N}(x,t)-u_k(x,t),\qquad k=0,1,2,\dots\ .$$

By using representation $u_0(x,t)$ (see (8)), we get

$$z_0(x,t)=[U_N(t,0)-U(t,0)]\varphi(x-at)+\int_0^t[U_N(t,s)-U(t,s)]f(x-a(t-s),s)ds.$$

For $z_0(x,t)$ estimation we have to evaluate $[U_N(t,0)-U(t,0)]$. Let us see at first $||T_N(t;B_{i,j})||$, by using inequalities obtained in [3]. We have

$$||T_N(t_j-t_{j-1};B_{i,j})||=||T_N(\tau;B_{i,j})||\le||T(\tau;B_{i,j})||+$$

$$+||T_N(\tau;B_{i,j})-T(\tau;B_{i,j})||\le1+Cexp\{-\delta(\tau N^2)^{1/3}\}.$$

Then if $t\in[t_{j-1},t_j],\quad s\in[t_{p-1},t_p]$ for the approximation $U_N(t,s)$ of the evolution operator we get

$$||U_N(t,s)||\le||T_N(t-t_{j-1};B_{i,j})[\prod_{k=1}^{j-p-1}T_N(\tau;B_{i,j-k})]T_N(t_p-s;B_{i,p})||\le$$

$$\le(1+C)^2\prod_{k=1}^{j-p-1}||T_N(\tau;B_{i,j-k})||\le(1+C)^2(1+Cexp\{-\delta(\tau N^2)^{1/3}\})^{N_2}\le$$

$$\le(1+C)^2exp(CN_2exp\{-\delta(\tau N^2)^{1/3}\}).\qquad(15)$$

Let us suppose that

$$N\ge N_2.\qquad(16)$$

Thus from (9) we have

$$||U_N(t,s)||\le(1+C)^2exp(CNexp\{-\delta(TN)^{1/3}\})\equiv M(N).\qquad(17)$$

By increasing N we can make the quantity of the right hand side in (17) as close to $(1+C)^2$ as we want. Further we'll look for an estimation for the approximation error of evolution operator:

$$||U_N(t,s)-U(t,s)|| = ||T_N(t-t_{j-1};B_{i,j})[\prod_{k=1}^{j-p-1} T_N(\tau;B_{i,j-k})]T_N(t_p-s;B_{i,p})-$$

$$-T(t-t_{j-1};B_{i,j})[\prod_{k=1}^{j-p-1} T(\tau;B_{i,j-k})]T(t_p-s;B_{i,p})|| =$$

$$||[T_N(t-t_{j-1};B_{i,j})-T(t-t_{j-1};B_{i,j})][\prod_{k=1}^{j-p-1} T(\tau;B_{i,j-k})]T(t_p-s;B_{i,p})+$$

$$T_N(t-t_{j-1};B_{i,j})[T_N(\tau;B_{i,j-1})-T(\tau;B_{i,j-1})][\prod_{k=2}^{j-p-1} T(\tau;B_{i,j-k})]T(t_p-s;B_{i,p})+$$

$$+\ldots+T_N(t-t_{j-1};B_{i,j})[\prod_{k=1}^{j-p-1} T_N(\tau;B_{i,j-k})][T_N(t_p-s;B_{i,p})-T(t_p-s;B_{i,p})]|| \leq$$

$$\leq (1+C)\sum_{k=0}^{j-p-1}(\prod_{r=1}^{k}||T_N(\tau;B_{i,j-r+1})||)||T_N(\tau;B_{i,j-k})-T(\tau;B_{i,j-k})|| \leq$$

$$\leq (1+C)\sum_{k=0}^{N_2}(1+Cexp\{-\delta(\tau N^2)^{1/3}\})^k Cexp(-\delta(\tau N^2)^{1/3}) =$$

$$= (1+C)Cexp(-\delta(\tau N^2)^{1/3})\frac{1-(1+Cexp\{-\delta(\tau N^2)^{1/3}\})^{N_2}}{-Cexp(-\delta(\tau N^2)^{1/3})} =$$

$$= [(1+Cexp\{-\delta(\tau N^2)^{1/3}\})^{N_2}-1](1+C).$$

It follows from the assumption (17) and above inequality that:

$$||U_N(t,s)-U(t,s)|| \leq (1+C)(exp(CNexp\{-\delta(TN)^{1/3}\})-1) = M_1(N). \tag{18}$$

Then

$$||z_0(x,t)|| \leq ||U_N(t,0)-U(t,0)||||\varphi(x-at)|| + \int_0^t ||U_N(t,s)-U(t,s)||\times$$

$$\times||f(x-a(t-s),s)||ds \leq M_1(N)(||\varphi(x-at)||+$$

$$+\int_0^t ||f(x-a(t-s),s)||ds) \le M_1(N)U_0(x,t).$$

One can get by using estimation for $U_N(t,s)$ and $U_N(t,s)-U(t,s)$ (17), (18), for $z_k(x,t)$ that

$$||z_k(x,t)|| \le \int_0^t ||U_N(t,s)||||B_0(x-a(t-s),s)-B(x-a(t-s),s)||\times$$

$$\times||z_{k-1}(x-a(t-s),s)||ds + \int_0^t ||U_N(t,s)-U(t,s)||||B_0(x-a(t-s),s)-$$

$$-B(x-a(t-s),s)||||u_{k-1}(x-a(t-s),s)||ds \le M(N)q\int_0^t ||z_{k-1}(x-a(t-s),s)||ds+$$

$$+M_1(N)q\int_0^t ||u_{k-1}(x-a(t-s),s)||ds) \le$$

$$\le M(N)q\int_0^t ||z_{k-1}(x-a(t-s),s)||ds + M_1(N)\frac{(qt)^k}{k!}U_0(x,t) \le \ldots \le$$

$$\le \frac{(M(N)qt)^k}{k!}M_1(N)U_0(x,t) + \frac{(qt)^k}{k!}M_1(N)U_0(x,t)\sum_{p=0}^{k-1}(qM(N))^p =$$

$$= \frac{(M(N)qt)^k}{k!}M_1(N)U_0(x,t)\left(1+\sum_{p=0}^{k-1}q^pM(N)^{p-k}\right).$$

Just as $p=\overline{0,k-1}$, so $M^{k-1}\ge M^p$, $\forall p$. And thus $M^{-k}\le M^{-p-1}$. That's why

$$\sum_{p=0}^{k-1}q^pM(N)^{p-k} \le M^{-1}\sum_{p=0}^{k-1}(q/M(N))^p.$$

The fulfillment of the following inequality can always be reached by reducing q:

$$\frac{q}{M(N)} < 1. \qquad (19)$$

And that's why

$$M^{-1}(N)\sum_{p=0}^{k-1}(q/M(N))^p \le M^{-1}(N)\sum_{p=0}^{\infty}(q/M(N))^p = \frac{1}{M(N)-q}.$$

So we have:

$$\frac{(M(N)qt)^k}{k!}M_1(N)U_0(x,t)\left(1+\sum_{p=0}^{k-1}q^pM(N)^{p-k}\right) \le \frac{(M(N)qt)^k}{k!}M_1(N)U_0(x,t)\times$$

$$\times\left(1+\frac{1}{M(N)-q}\right) = \frac{(M(N)qt)^k}{k!}M_1(N)U_0(x,t)C_1.$$

Then

$$||z_k(x,t)|| \le \frac{(M(N)qt)^k}{k!}M_1(N)U_0(x,t)C_1. \tag{20}$$

Let us estimate the full approximation error of FD-method:

$$||u(x,t)-u_N^{(m)}(x,t)|| \le ||u(x,t)-u^{(m)}(x,t)|| + ||u^{(m)}(x,t)-u_N^{(m)}(x,t)|| \le$$

$$\le ||u(x,t)-u^{(m)}(x,t)|| + \sum_{k=0}^{m}||z_k(x,t)|| \le$$

$$\le \frac{(qt)^{m+1}}{(m+1)!}e^{qt}U_0(x,t) + \sum_{k=0}^{m}\frac{(M(N)qt)^k}{k!}U_0(x,t)M_1(N)C_1 \le$$

$$\le \frac{(qt)^{m+1}}{(m+1)!}e^{qt}U_0(x,t) + M_1(N)U_0(x,t)C_1exp(M(N)qt) =$$

$$U_0(x,t)[\frac{(qt)^{m+1}}{(m+1)!}e^{qt} + M_1(N)C_1exp(M(N)qt)]. \tag{21}$$

So we can formulate the following statement:

Theorem 2 *Let conditions of the theorem 1 and inequality (19) be valid. The solution of problem (2) is found by the following algorithm*

$$u_{0,N}(x,t) = U_N(t,0)\varphi(x-at) + \int_0^t U_N(t,s)f(x-a(t-s),s)ds,$$

$$u_{k,N}(x,t) = \int_0^t U_N(t,s)[B_0(x-a(t-s),s)-B(x-a(t-s),s)]u_{k-1,N}(x-a(t-s),s)ds,$$

$$k = 1, 2, \ldots, m,$$

$$u_N^{(m)} = \sum_{k=0}^{m} u_{k,N}(x,t).$$

Then the algorithm has an exponential rate of convergence and estimation (21) is valid.

Proof. It remains to show that estimation (21) has an exponential rate of convergence.

Indeed, if $m = N$, then it easy to check that the first part of the estimation (21) has an exponential rate of convergence. Let us show that the second part of the estimation (21) has an exponential rate of convergence too. We have

$$\lim_{N\to\infty} M_1(N)N^k = \lim_{N\to\infty} (1 + C)(exp(CNexp\{-\delta(TN)^{1/3}\}) - 1)N^k.$$

It's obviously that $e^x - 1 \le xe^x$. So we get

$$(exp(CNexp\{-\delta(TN)^{1/3}\})-1) \le CNexp\{-\delta(TN)^{1/3}\}exp(CNexp\{-\delta(TN)^{1/3}\}).$$

Thus one can get

$$\lim_{N\to\infty} M_1(N)N^k = \lim_{N\to\infty} CN^{k+1}exp\{-\delta(TN)^{1/3}\}exp(CNexp\{-\delta(TN)^{1/3}\}) = 0,$$

$$\forall k \in N.$$

This completes the proof.

□

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